ω_3 does control the breakdown of smooth solutions of the 3D compressible equations as will be seen in this study.

The expression $\vec{f} \cdot \Psi$ in Equation (2.15) will vanish further below when we take a dot product in the z^* direction of flow or $\vec{k}$ direction downstream in tube. We therefore do not include it after Equation(2.24) below. Upon multiplication of Equation (2.17) by,

$$H = \frac{\rho\vec{b} \cdot \vec{f}}{\frac{\partial\rho}{\partial t}} \tag{2.22}$$

the resulting equation is

$$\begin{aligned}\left[\rho^2 H \tfrac{\partial}{\partial t}(\nabla \cdot \vec{b}) + \rho^2 \vec{b} \cdot \vec{f} \nabla \cdot \vec{b} + H\vec{b} \cdot \nabla(\rho \tfrac{\partial\rho}{\partial t})\right] + H\nabla \cdot (\vec{b}\rho^2 \nabla \cdot \vec{b}) = \\ \mu H \nabla \cdot \nabla^2 \vec{b} + \tfrac{\mu}{3} H \nabla \cdot (\nabla(\nabla \cdot \vec{b})) + H\nabla \cdot (\nabla P) + H\nabla \cdot \rho \vec{F_T}\end{aligned} \tag{2.23}$$

which results upon using Equation (2.15) in,

$$\begin{aligned}\rho^2 H \tfrac{\partial}{\partial t}(\nabla \cdot \vec{b}) - \vec{f} \cdot \left(\rho^2 \tfrac{\partial \vec{b}}{\partial t} + \rho\vec{b}\tfrac{\partial\rho}{\partial t}\right) - \left\|\vec{f}\right\|^2 + \vec{f} \cdot \nabla^2 \vec{b} + \tfrac{\mu}{3} \vec{f} \cdot \nabla(\nabla \cdot \vec{b}) - \vec{f} \cdot \nabla P + H\vec{b} \cdot \nabla(\rho \tfrac{\partial\rho}{\partial t}) + \\ H\nabla \cdot (\vec{b}\rho^2 \nabla \cdot \vec{b}) = \mu H(\nabla \cdot \nabla^2 \vec{b}) + \tfrac{\mu}{3} H\nabla \cdot (\nabla(\nabla \cdot \vec{b})) + H\nabla \cdot (\nabla P) + H\nabla \cdot \rho\vec{F_T} + \Psi\end{aligned} \tag{2.24}$$

The continuity equation is written in terms of $\vec{b}$ as,

$$\frac{\partial\rho}{\partial t} + \rho\nabla \cdot \vec{b} + \vec{b} \cdot \nabla\rho = 0 \tag{2.25}$$

and

$$\nabla \cdot \vec{b} = -\frac{1}{\rho}\frac{\partial\rho}{\partial t} - \frac{1}{\rho}\nabla\rho \cdot \vec{b} \tag{2.26}$$

where the following compact expression is given,

$$Y^* = \nabla \cdot \vec{b} \tag{2.27}$$

For the term $\rho\vec{f} \cdot \vec{b}\frac{\partial\rho}{\partial t}$ in Equation (2.24) we obtain upon using Equations (2.15) again, (2.26) and multiplying by $(\vec{f} \cdot \vec{f})^{-1}$ in Equation (2.24) and, using properties of third derivatives involving the gradient and in particular the fact that the Laplacian of the divergence of a vector field is equivalent to the divergence of the Laplacian of a vector field, leads to the following form,

$$\begin{aligned}W^* \tfrac{\partial Y^*}{\partial t} - G(\rho, \tfrac{\partial\rho}{\partial t})W^* - F(\rho, \tfrac{\partial\rho}{\partial t})\vec{b}\vec{f}(1 + \vec{f} \cdot \nabla P) - \rho^{-2}V(\mu)W^*\nabla^2(Y^*) - 2\vec{b}\vec{f}\tfrac{\partial\rho}{\partial t}\tfrac{\rho^{-1}}{\left\|\vec{f}\right\|^2}\vec{f} \cdot \left(\tfrac{\partial\vec{b}}{\partial t}\right) + \Omega + \\ 2\vec{b}\vec{f}\tfrac{\partial\rho}{\partial t}\tfrac{\rho^{-3}}{\left\|\vec{f}\right\|^2}\mu\vec{f} \cdot \nabla^2\vec{b} + 2\vec{b}\vec{f}\tfrac{\partial\rho}{\partial t}\tfrac{\rho^{-3}}{\left\|\vec{f}\right\|^2}\tfrac{\mu}{3}\vec{f} \cdot \nabla(\nabla \cdot \vec{b}) - \rho^{-2}\vec{b}\vec{f}\tfrac{1}{\left\|\vec{f}\right\|^2}\vec{b} \cdot \vec{f}\,\nabla \cdot \left(\vec{b}\rho^2\nabla \cdot \vec{b} + \nabla P + \rho\vec{F_T}\right) = 0\end{aligned} \tag{2.28}$$

where $\Omega = \rho^{-2}H\vec{b} \cdot \nabla(\rho\frac{\partial\rho}{\partial t})$ in Equation (2.28).

We consider Ω term now. We use Equation (2.15) and the following identities,

$$\vec{b} \cdot \nabla(\nabla^2\vec{b}) = \frac{1}{2}\nabla(\nabla^2\vec{b} \cdot \nabla^2\vec{b}) - \nabla^2\vec{b} \times (\nabla \times \nabla^2\vec{b})$$

$$\vec{b} \cdot \nabla(\rho^2\frac{\partial\vec{b}}{\partial t}) = \nabla(\frac{b^2}{2}) - \vec{b} \times \nabla \times (\rho^2\frac{\partial\vec{b}}{\partial t})$$

Further in this paper we will take curl of the desired equation and the curl of the previous expression above when dotted with $\vec{b}$ will be zero. As a result the term that is left over is,

$$L_1 = \nabla \times (2\nabla^2\vec{b} + \nabla^2\vec{b} \times \nabla^2\vec{\omega})$$

If the Laplacian of the vorticity vector is the triple (-2,-2,-2)(ie. vorticity is quadratic), then using the following identity,

$$\nabla \times (\vec{a} \times \vec{s}) = \vec{a}(\nabla \cdot \vec{s}) - \vec{s}(\nabla \cdot \vec{a}) + (\vec{s} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{s},$$

$$L_1 = 0$$

Next,

$$W^* = \left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{f}\right)\vec{b} = \left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{f}\right)\cdot\vec{b} + \left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{f}\right)\times\vec{b} = \xi + \vec{f} \times \left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{b}\right) \quad (2.29)$$

This involves the vector projection of $\vec{b}$ onto $\vec{f}$ which is written in the conventional form,

$$\mathrm{proj}_{\mathbf{f}}\,\mathbf{b} = \frac{\mathbf{f}\cdot\mathbf{b}}{||\mathbf{f}||^2}\mathbf{f} \quad (2.30)$$

Equation (2.28) can be written compactly as

$$\frac{\partial Y^*}{\partial t} - G(\rho, \frac{\partial \rho}{\partial t}) - \rho^{-2}V(\mu)\nabla^2 Y^* - \rho^{-2}\nabla^2 P = \frac{U_{\vec{f}}\left[\mathbf{Q}(\rho, \frac{\partial\rho}{\partial t}, \vec{b}, \frac{\partial\vec{b}}{\partial t}, \nabla P, \vec{F_T}) + \vec{f}\right]}{U_{\vec{f}}\vec{b}} \quad (2.31)$$

where $U_{\vec{f}}\vec{\xi}$ is the scalar projection for $\vec{b}$, $G = \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial t}\right)^2$, $\mathbf{Q}$ (a differential operator defined by Equations (2.16) and (2.28) and hence for a constant positive function α,

$$\frac{\partial Y^*}{\partial t} - G(\rho, \frac{\partial \rho}{\partial t}) - \rho^{-2}V(\mu)\nabla^2 Y^* - \rho^{-2}\nabla^2 P = \frac{\left\|\mathbf{Q}(\rho, \frac{\partial\rho}{\partial t}, \vec{b}, \frac{\partial\vec{b}}{\partial t}, \nabla P, \vec{F_T}) + \vec{f}\right\|}{\left\|\vec{b}\right\|} = \alpha \geq 0 \quad (2.32)$$

with solution in terms of a function $\mathcal{B}$,

$$Y^* = \nabla \cdot \vec{b} = \mathcal{B}(\alpha, r, \theta, z, t) \quad (2.33)$$

At this stage of the analysis we introduce the vorticity equation for compressible flow,

$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{a}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{a} - \vec{\omega}(\nabla\cdot\vec{a}) + \frac{\nabla\rho}{\rho^2} \times \nabla P + \nabla \times (\frac{\nabla\cdot\tau}{\rho}) + \nabla \times (\frac{F}{\rho})$$

We consider the third component of the vorticity equation in z^*. It is assumed that the vorticity is an exponential function of z^* and t^* , ie $\omega = r^* G(\theta^*)\mathrm{e}^{-\tanh(\alpha z^*)}\mathrm{e}^{-\tanh(\alpha t^*)}$, for some general function of θ^*. Recall that $a_1 = \rho b_1 = 0$ on surface $r^{*2} - \theta^{*2} - z^{*2} = 0$, $a_2 = \rho b_2 = \frac{1}{2r^*}\omega(r^*, \theta^*, z^*, t^*)(r^{*2} + \theta^{*2} + z^{*2})$. This can be non-zero except at the center of the tube where there the flow is irrotational, also $G(\theta^*) \neq 0$ further away from the center of tube. Substitution of this form of $\vec{\omega}$ into the vorticity equation and