Perturbative/Non-Perturbative Quantum Field Theory

Subjects: Others

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In this review, we focused on the foundations of quantum field theory, which is still believed to be the most fundamental theory, describing in principle all phenomena observed in atomic and particle physics. Unlike quantum mechanics, however, its foundations are still not cleared up. We attempted to describe how some novel approaches lead to a unified picture, in spite of the fact that several difficult open problems remain.

quantum fields divergent series Casimir effect Wightman axioms singularity hypothesis

1. Introduction

One of the most impressive measurements in physics is of the electron anomalous magnetic moment of the electron, which is known with a precision of a few parts in1014[1]. On the other hand, the theoretical prediction needed to match this result requires the use of perturbation theory containing up to five loop contributions. As is well known, higher order contributions in systems containing an infinite number of degrees of freedom lead to the appearance of divergences which are taken care of by renormalization theory [2]. Therenormalizedperturbation series in enormously successful theories, such as quantum electrodynamics (qed), is, however, strongly conjectured to be adivergentseries (see Reference [3] for the first result in this direction), and G. 't Hooft [4] has, among others, raised the question whether qed has a meaning in a rigorous mathematical sense.

Several years ago, Arthur S. Wightman, one of the leaders of axiomatic quantum field theory, asked the related question "Should we believe in quantum field theory?" , in a still very readable review article [5]. After his program of constructing interacting quantum field theories in four space-time dimensions failed, the question remains in the air, and we should like to suggest in the present review that the answer to it does remain affirmative, although the open problems are very difficult.

One of the problems with quantum field theory (qft) is that it seems to have lost contact with its main object of study, viz. , explaining the observed phenomena in the theory of elementary particles. One instance of this fact is that, except for a few of the lightest particles, all the remaining ones are unstable, and there exists up to the present time no single mathematically rigorous model of an unstable particle (this is reviewed in Reference [6], and we come back to this point Alternative theories have not supplied any new experimentally measured numbers in elementary particle physics, such as the impressive one in qed mentioned above.

A recent, important field of applications of qft came from condensed matter physics: the structure of graphene, of which certain (possibly experimentally verifiable) phenomena have been successfully studied by Gavrilov, Gitman et al. [7] with nonperturbative methods from the theory of strong-field qed with unstable vacuum. So far, however, the only experimental consequence of a non-perturbative quantum field theoretic model concerns the Thirring model [8] in its lattice version, the Luttinger model. The latter's first correct solution was due to Lieb and Mattis [9]; the rigorous explanation of the thereby new emergent quasi-particles through fluctuation observables was provided by Verbeure and Zagrebnov in Reference [10].

2. General Aspects of Nonperturbative Quantum Fields: Wightman Axioms for Interacting Quantum Fields, Dressed Particles in a Charged Sector, and Unstable Particles

We shall try to keep with the general objective of remaining at the theoretical physicist's mathematical level [2][11]. The theory will be defined by its vacuum expectation values (VEV) orn-point Wightman functionsof a (for simplicity) scalar fieldA(x):Wn(x1,⋯,xn)=(Ω, Because of the previously mentioned singular nature of the quantum fieldsA(x), theA(x)are not (operator-valued) functions of the space-time variablesx=(x0,x→), but ratherfunctionals, denoted byA(f), which may be heuristically pictured as "smeared" objects∫dxf(x)A(x), withfsmooth, fast decreasing at infinity functions, taken to belong to the Schwartz space [12]: one speaks of "operator-valued tempered distributions" on the Schwartz spaceS.

In Reference [13], pp. 107–110, it is shown that, if the (n-point)Wightman functionssatisfy(a.)the relativistic transformation law;(b.)the spectral condition;(c.)hermiticity;(d.)local commutativity;(e.)positive-definiteness,

then they are thevacuum expectation values of a field theorysatisfying the so-called Wightman axioms, except, eventually, the uniqueness of the vacuum state. We refer to Reference [13] for an account of Wightman theory; also see Reference [14].

We shallassume (a.)−(e.)for then-point functions of the observable fields, with, in addition, the following requirement:(f.)interacting fields are assumed to satisfy the singularity hypothesis (the forthcoming Definition).

In a non-perturbative framework, there exists a theory of renormalization of masses and fields, which "has nothing directly to do with the presence of infinities" (Reference [2], p. 441, Section 10.3). We adopt a related proposal, which we formulate as previously mentioned, for simplicity, for a theory of a self-interacting scalar fieldAof massmsatisfying the Wightman axioms, assumed to be an operator-valued tempered distribution on the Schwartz spaceS.

The following result is the spectral representation of the two-point functionW2(Reference [2], p. 457): (103)W2m(x−y)=⟨Ω,A(x)A(y)Ω⟩=1i∫0∞dρ(m∘2)Δ+m∘(x−y), whereΩdenotes the vacuum vector,x=(x0,x→), and (104)Δ+m∘(x)=i2(2π)3∫R3d3k→e−ix0m∘2+k→2+ix→·k→m∘2+k→2 is the two-point function of the free scalar field of massm∘. It is further assumed that (105)⟨Ω,A(f)Ω⟩=0∀f∈S. The spectral measureρmay be further decomposed in the sum of a discrete and a continuous part (106)$d\rho(m_\circ^2)=Z\delta(m_\circ^2-m^2)+d\sigma(m_\circ^2)$, where$\sigma$is a continuous measure and

Of course, if$Z=0$in (106), there is no discrete component of mass$m$in the total mass spectrum of the theory. The positivity of$Z$is due to the positive-definiteness conditions e.) See Reference [11], p. 50, or (Reference [2],

One of the most important features of relativistic quantum field theory is the behavior of the theory at large momenta (or large energies). Renormalization group theory [15] has contributed a significant lore to this issue (even if none of it has been made entirely rigorous): it strongly suggests that the light-cone singularity of the two-point functions of interacting theories is stronger than that of a free theory: this is expected even in asymptotically free quantum chromodynamics, where the critical exponents are anomalous. We refer to this as the "singularity hypothesis", which will be precisely stated in the next section. This hypothesis is also verified in each order of perturbation theory (if the interaction density has engineering dimension larger than 2).

In order to formulate the singularity hypothesis in rigorous terms, the Steinmann scaling degree$sd$of a distribution [16] is a natural concept: for a distribution$u\in S'(R^n)$, let$u_\lambda$denote the "scaled distribution", defined by $u_\lambda(f)\equiv u(f(\lambda^{-1}\cdot))$. As$\lambda\to 0$, we expect that$u_\lambda\approx\lambda^{-\omega}$for some$\omega$, the "degree of singularity" of the distribution$u$. (108)$sd(u)\equiv\inf\{\omega\in R \mid \lim_{\lambda\to 0}\lambda^\omega u_\lambda=0\}$, with the proviso that, if there is noωsatisfying the limiting condition above, we set$sd(u)=\infty$. For the free scalar field of mass$m\geq 0$, it is straightforward to show from the explicit form of the two-point function in terms of modified Bessel functions that (109)$sd(\Delta_+)=2$ (see, e.g., Reference [11], (5.15)).

We say that the singularity hypothesis holds for an interacting scalar field if (110)$sd(W_+)>2$.

In Reference [17], it was proved that:

If the total spectral mass is finite, i.e.,(111)$\int_0^\infty d\rho(a^2)<\infty$,then(112)$sd(W_+)\leq 2$,i.e., the scaling degree ofW_+cannot be strictly greater than that of a free theory; thus, by Definition 1, the singularity hypothesis(110)is not satisfied.

The singularity hypothesis holds for an interacting scalar field only if$\int_0^\infty d\sigma(m_\circ^2)=\infty$. This necessary condition is independent of the value of$0\leq Z<\infty$.

The importance of the above theorem, and especially of its corollary is that it provides a mathematical foundation for the forthcoming interpretation of the condition (113)$Z=0$. In this sense, it is a complement to the foundations of quantum field theory. In order to understand why this is so, we have to make a brief interlude.

For the purposes of identification with Lagrangian field theory, one may equate the$A(.)$of (103) with the "bare" scalar fieldϕ_B(Reference [2], p. 439), whereby (114)$A=\sqrt{Z}A_{phys}$, under the condition (115)$Z>0$. Under the same condition (115), the assumption of equal time commutation relations (ETCR) for the physical fields may be written (in the distributional sense) (116)$\left[\frac{\partial A_{phys}(x_0,\vec{x})}{\partial x_0},A_{phys}(x_0,\vec{y})\right]=-\frac{i}{Z}\delta(\vec{x}-\vec{y})$. Together with (103) and (116), one obtains (Reference [11], (9.19), Reference [2], (10.7.18) (suitably modified by the factor$\frac{1}{Z}$)): (117)$\frac{1}{Z}=\int_0^\infty d\rho(m_\circ^2)$.

Formula (117) has been extensively used as a heuristic guide, even, for instance, by the great founders of axiomatic (or general) quantum field theory, Wightman and Haag. Indeed, in Reference [18], p. 201, it is observed that “$\int_0^\infty d\rho(m_\circ^2)=\infty$is what is usually meant by the statement that the field-strength renormalization is infinite”. This means that the fields in an interacting theory are more singular objects than in the free theory, and we do not have the ETCR.” Both assertions seems to substantiate the conjecture that$Z=0$is expected to be ageneralcondition for interacting fields.

If (117) holds, only$Z=0$is compatible with the singularity hypothesis.

It follows from Corollarys 1 and 2 that the two definitions of$Z$, in (106) and in (117), arenotequivalent. Since the ETCR, which implies (117), is not generally valid for interacting fields, as briefly reviewed in the forthcoming paragraph, we conclude that the singularity hypothesis opens the possibility of the non-universal validity of (113).

The hypothesis of ETCR has been in serious doubt for a long time, see, e.g., the remarks in Reference [13], p. 101. Its validity has been tested [19] in a large class of models in two-dimensional space-time, where it was definitely proved not to hold in the case of the Thirring model [8] for large coupling. The only case found in [19] in which the ETCR holds was the Schwinger model, which is known to be equivalent to a canonical theory of a massive vector field! A nother interesting case of canonical behavior were the canonical interacting quantum fields on two-dimensional de Sitter space [20], where, however, the canonical character was intrinsically due to the geometry of de Sitter space.

Unfortunately, however, the ETCR (for interacting fields!) is still assumed to hold, without comment, in standard treatises ([2], p. 460; [11], p.47).

Although, when0<z<∞,zis interpreted="" as="" the="" non-perturbative="" field="" strength="" renormalization,="" relating="" “bare”="" fields="" to="" physical="" fields,="" in="" (114),="" remaining="" case="" (113)="" remains="" be="" understood.="" stated="" reference="" [<a="" class="html-bibr" href="https://www.mdpi.com/2218-1997/7/7/229/htm#B2-universe-07-00229">2], p. 461, “the limit$Z=0$has an interesting interpretation as a condition for a particle to be composite rather than elementary”. This brings us to our next topic. </z<∞,zis>

3. A Proposal for the Meaning of the Condition Z=0: The Presence of Massless and Unstable Particles

In the presence of massless particles (photons), Buchholz [21] used Gauss’ law to show that the discrete spectrum of the mass operator (118)$P_\sigma P^\sigma=M^2=$ Above,$P_0$is the generator of time translations in the physical representation, i.e., the physical hamiltonian$H$, and$\vec{P}$is the physical momentum. This fact is interpreted as a confirmation of the phenomenon that particles carrying an electric charge are accompanied by clouds of soft photons.

Buchholz formulates adequate assumptions which must be valid in order that one may determine the electric charge of a physical state$\Phi$with the help of Gauss’ law:(119)$\langle\Phi,j_\mu\Phi\rangle=\langle\Phi,\partial^\nu F_{\nu,\mu}\Phi\rangle$. $F_{\nu,\mu}$denotes the electromagnetic field observable, and (119) is assumed to hold in the sense of distributions onS(Rn).

When endeavoring to apply Buchholz's theorem to concrete models, such asqed1+3, problems similar to those occurring in connection with the charge superselection rule [22] arise. The most obvious one is that Gauss' law (119) is only expected to be valid (as an operator equation in the distributional sense) in non-covariant gauges, the Coulomb gauge in the case ofqed1+3, but not in covariant gauges [22]. If we adopt the present framework, our option is to use the Coulomb gauge and to define the theory in terms of then-point Wightman functions of observable fields, i.e., gauge-invariant fields, thus maintaining Hilbert-space positivity. β∫0∞dρ2(m∘2)Δ+(x−y;m∘2), withdρph,dρ1,dρ2positive, measures, andρ1satisfying certain bounds with respect

Forqed1+3in the Coulomb gauge, assuming it exists in the sense of the framework of this section and satisfies the assumptions of Buchholz's theorem, the following condition holds:

Z 2= 0 (122)

Above,Ψdenote observable fermion fields, which we assume to exist as a generalization of those constructed by Lowenstein and Swieca [23] inqed1+1; also see Reference [24] for a similar attempt in perturbativeqed1+3. In the words of Lieb and Loss [25], who were the first to observe this phenomenon in a relativistic model of qed, "the electron Hilbert space is linked to the photon Hilbert space in an inextricable way". Thereby, in this way, "dressed photons" and "dressed electrons" arise as new entities.

We now come back to Weinberg's suggestion that the conditionZ=0describes unstable particles.

Turning to scalar fields for simplicity, we consider the case of a scalar particleC, of massmC, which may decay into a set of two (for simplicity) stable particles, each of massm. We have energy conservation in the rest frame andq→ithe momenta of the two particles in the rest frame ofC:(123)mC>2m. In order to stress the practical effects of the confusion of the two definitions ofZin the literature, when the ETCR is assumed (including [2]), it should be remarked that the first reference given by Weinberg on a model for unstable particles [26] assumes (what ammounts to)mC≤2minstead of (123)

In order to check thatZ=ZC=0when (123) holds, while0<z<∞is valid="" in="" the="" stable="" casemc<2m,="" a="" model,="" we="" are="" beset="" with="" difficulty="" to="" obtain="" information="" on="" two-point="" function.="" <="" p=""></z<∞is>

There exists a quantum model of Lee type of a composite (unstable) particle, satisfying (123), where (113) was indeed found, that of Araki et al. [27]. Unfortunately, however, the (heuristic) results in Reference [27] have one major defect: their model contains "ghosts". A very good review of the existent (nonrigorous) results on unstable particles is the article by Landsman [28], to which we refer for further references and hints on the intuition behind the criterion (113).

In the next section, we come back to a set of models for atomic resonances and particles, which might support the suggested picture of quantum field theory in terms of "dressed" and unstable particles, and, in the last section, we discuss the crucial conceptual issues and difficulties associated with this program, comparing it with alternative approaches.

4. Models for Atomic Resonances, Unstable and "Dressed" Particles: What Distinguishes Quantum Field Theory from Many-Body Systems?

In Reference [6] the model below—the Lee-Friedrichs model of atomic resonances—was revisited. onL2(R3)(see, e.g., Reference [29]), which describes the photons. Then, the one-dimensional subspaceP0Hconsists of the ground state vector (132)Φ0≡|−)⊗|Ω), with energy zero, where (133)σz|±)=±|±) denote the upper|+)and lower|−)atomic levels, and|Ω)denotes the zero-photon state inF. Note thatΦ0is also eigenstate of the free HamiltonianH0, with energy zero, and we say, therefore, that the model has a persistent zero particle state.

We shall refer to the model described by (124) asModel 1. with the further correspondences |+)=V+|0),|−)=N+|0), where|0)denotes the fermion no-particle state, Model 1 becomes the usual Lee model for particles, and Model 2 a "refined" Lee model for particles; we refer to them asModel 3andModel 4, respectively. There is one big difference, however, between the atomic resonance case and the particle case. To the particle versions, Model 3 and Model 4, assuming they are well-defined, the forthcoming framework is applicable.

Let a theory of a scalar field of massm>0be invariant under the euclidean group, that is, the group of translations and rotations of euclidean spacex→→Rx→+a→, whereRdenotes a rotation. By Haag's theorem (Reference [18], p. 249), in a euclidean field theory which uses the Fock representation, the no-particle stateΨ0is euclidean invariant, i.e., (135)U(a→,R)Ψ0=Ψ0. We have the following:

Let the Hamiltonian be of the form(136)H=∫H(x→)dx→,whereH(x→)satisfies(137)U(a→,1)H(x→U(a→,1)−1=H(x→+a→).Then, ifΨis any state invariant underU(a→,1) , i.e., (138)U(a→,1)Ψ=Ψ, thenΨbelongs to the domain of H only ifHΨ=0.

We have that(Ψ,H(x→)H(y→)Ψ)depends only onx→−y→, so that (139)||HΨ||2=∫∫dx→dy→(Ψ,H(x→)H(y→)Ψ)=0or∞, according to whetherHΨ=0or||HΨ||=∞, that is,Hmust annihilate any translation-invariant state to which it is applicable. □

This result may be applied to Model 3 (assuming the renormalizations performed such that the pointwise limit can be taken, leading to an euclidean invariant quantum field theory), with the no-particle state identified to the state given byΨ≡|Ω)⊗|−), in and|Ω)the no-particle photon state: We say that there is novacuum polarization. On the other hand, for Model 4,Ψdoes not belong to the domain ofHdue to the termσ+a†(g)in (134), which, in terms of fermion operators equals,V+Na†(g), andV+N|−)=|+). In the case there is vacuum polarization, Theorem 3 implies that there exists an "infinite energy barrier" between the Fock no-particle state and the true vacuum: non-Fock representations are required,

All this being said, Model 3 turns out, very unexpectedly, to be afflicted by "ghosts", i.e., states of negative norm (see Reference [11], Chapter 12). We say unexpectedly because the occurrence of "ghosts" in relativistic quantum field theory is known to be a consequence of the use of manifestly covariant gauges (see Reference [15]), and the Lee model is not relativistically invariant. One may consider, however, the model with recoil, describing the interaction between the photon field andtwoparticlesVandN, with energiesEV(p)=(M2+p2)1/2andEN(p)=(m2+p2)1/2, and interaction energyHI=∫d3pd3kg(p,k)[V † (p)N(p−k)a(k)+h.c.]λ, withλproportional to the charge, andg(p,k)≡f(p,k)[8EV(p)EN(p−k)ω(k)]−1/2,ω(k)=(μ2+k2)1/2, the latter representing the (eventually massive) "photon" energy.

We henceforth refer to the Yndurain versions of models 3 and 4 as Model 3Y and Model 4Y.

Model 2 (for zero temperature, as a model of atomic resonances) may be the simplest prototype of a model with vacuum polarization. Model 3Y should be suitable to study the criterionZ=0for unstable particles. The subtlest point in this connection is the fact that the term proportional to the delta measure in (106) has coefficientZ, and, in its absence, due to the conditionZ=0, the (renormalized) mass seems to remain undetermined. One must, therefore, be able to determine the renormalized mass uniquely fromalternativegeneral conditions on the Hamiltonian, such as the requirement that it be self-adjoint and bounded below.

We close this section with some remarks of what distinguishes quantum field theory from many-body systems. In contrast, quantum field theories display in general vacuum polarization, as typified by Model 4Y, which is an approximation of the usual trilinear coupling terms which occur in qed and also in quantum chromodynamics (qcd), as a consequence of relativistic invariance and the gauge principle (see References [2][15]). Coupled with Haag's theorem and Theorem 3, this implies non-Fock representations, as previously discussed. What we wish to emphasize is that, in the case of quantum field theories, such representations arise due to a particular reason, namely vacuum polarization.

References

1. Zyla, P.A.; Barnett, R.M.; Beringer, J.; Dahl, O.; Dwyer, D.A.; Groom, D.E.; Lin, C.-J.; Lugovsky, K.S.; Pianori, E.; Robinson, D.J.; et al. Review of Particle Physics. Prog. Theor. Exp. Phys. 2020, 8, 083C01.
2. Weinberg, S. The Quantum Theory of Fields Volume I—Foundations; Cambridge University Press: Cambridge, UK, 1995.
3. Thirring, W. On the divergence of perturbation theory for quantized fields. Helv. Phys. Acta 1953, 26, 33.

4. Hooft, G. Reflections on the renormalization procedure for gauge theories. Nucl. Phys. B 2016, 912, 4.
5. Wightman, A.S. Should we believe in quantum field theory? In The Whys of Subnuclear Physics; Zichichi, A., Ed.; Plenum Press: New York, NY, USA, 1979.
6. Wreszinski, W.F. Unstable states in a model of nonrelativistic quantum electrodynamics. J. Stat. Phys. 2021, 182, 26.
7. Gavrilov, S.P.; Gitman, D.M.; Dmitriev, V.D.; Panferov, A.D.; Smolyansky, S.A. Radiation problems accompanying carrier production by an electric field in graphene. Universe 2020, 6, 205.
8. Thirring, W.E. A soluble relativistic field theory. Ann. Phys. 1958, 3, 91.
9. Mattis, D.C.; Lieb, E.H. Exact solution of a many-Fermion system and its associate Bose field. J. Math. Phys. 1965, 6, 304–312.
10. Verbeure, A.; Zagrebnov, V.A. About the Luttinger model. J. Math. Phys. 1993, 34, 785.
11. Barton, G. An Introduction to Advanced Field Theory; Interscience: Hoboken, NJ, USA, 1963.
12. Schwartz, L. Mathematics for the Physical Sciences; Dover Publ. Inc.: Mineola, NY, USA, 2008.
13. Streater, R.F.; Wightman, A.S. PCT, Spin and Statistics and All That; W. A. Benjamin, Inc.: Princeton, NJ, USA, 1964.
14. Wightman, A.S. Quantum field theory in terms of vacuum expectation values. Phys. Rev. 1956, 101, 860.
15. Weinberg, S. The Quantum Theory of Fields Volume 2; Cambridge University Press: Cambridge, UK, 1996.
16. Steinmann, O. Perturbation Expansions in Axiomatic Field Theory. Lect. Notes. Phys. 1971, 11.
17. Jaekel, C.D.; Wreszinski, W.F. A criterion to characterize interacting theories in the Wightman framework. Quantum Stud. Math. Found. 2021, 8, 51.
18. Wightman, A.S. Introduction to some aspects of quantized fields. In High Energy Electromagnetic Interactions and Field Theory; Lévy, M., Ed.; Gordon and Breach: Philadelphia, PA, USA, 1967.
19. Wreszinski, W.F. Equal time limits of anticommutators in relativistic theories. Nuovo. Cim. A 1971, 1, 691–705.
20. Jaekel, C.D.; Mund, J. Canonical interacting quantum fields in two-dimensional de Sitter space. Phys. Lett. B 2017, 772, 786–790.
21. Buchholz, D. Gauss' law and the infraparticle problem. Phys. Lett. B 1986, 174, 331.
22. Strocchi, F.; Wightman, A.S. Proof of the charge superselection rule in local relativistic quantum field theory. J. Math. Phys. 1974, 15, 2198.

23. Lowenstein, J.H.; Swieca, J.A. Quantum electrodynamics in two dimensions. Ann. Phys. 1971, 68, 172–195.
24. Steinmann, O. Perturbative QED in terms of gauge invariant fields. Ann. Phys. 1984, 157, 232–254.
25. Lieb, E.H.; Loss, M. Stability of a model of relativistic quantum electrodynamics. Comm. Math. Phys. 2002, 228, 561.
26. Houard, J.; Jouvet, B. Étude d’un modèle de champs quantifiés à constante de renormalisation nulle. Nuovo Cim. 1960, 18, 466.
27. Araki, H.; Munakata, Y.; Kawaguchi, M.; Goto, T. Quantum field theory of unstable particles. Progr. Theor. Phys. 1957, 17, 419.
28. Landsman, N.P. Non-shell unstable particles in thermal field theory. Ann. Phys. 1988, 186, 141–205.
29. Martin, P.; Rothen, F. Many Body Problems and Quantum Field Theory, 2nd ed.; An Introduction; Springer: Berlin/Heidelberg, Germany, 2004.