

Perfect Division

Subjects: Others

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An exact division, also called even division or consensus division, is a division of a heterogeneous resource ("cake") to several subsets such that each of n people with different tastes agree about the valuations of the pieces.^{[1]:127} For example, consider a cake which is half chocolate and half vanilla. Alice values only the chocolate and George values only the vanilla. The cake is divided to three pieces: one piece contains 20% of the chocolate and 20% of the vanilla, the second contains 50% of the chocolate and 50% of the vanilla, and the third contains the rest of the cake. This is a consensus division, as both Alice and George value the three pieces as 20%, 50% and 30% respectively. As the example illustrates, a consensus division is not necessarily fair. For example, if the 20% piece is given to Alice and the 50% is given to George, this is obviously unfair to Alice. In the theory of cake, consensus divisions are often used as subroutines for creating fair divisions. Consensus divisions always exist, but they cannot be found by discrete protocols (with a finite number of queries). In some cases, exact divisions can be found by moving-knife protocols. Near-exact divisions can be found by discrete protocols.

Keywords: chocolate ; subroutines ; moving-knife

1. Definitions

Let w_1, w_2, \dots, w_k be k weights whose sum is 1. Assume that all n partners value the cake C as 1.

An **exact division** (aka **consensus division**) in the ratios w_1, w_2, \dots, w_k is a partition of the cake to k pieces: $C = X_1 \sqcup \dots \sqcup X_k$, such that for every partner i and every piece j :

$$V_i(X_j) = w_j$$

I.e., there is a consensus among all partners that the value of piece j is exactly w_j .^{[1]:127}

1.1. Near-Exact Division

For every $\varepsilon > 0$, An **ε -near-exact division** in the ratios w_1, w_2, \dots, w_k is a division in which:

$$|V_i(X_j) - w_j| < \varepsilon$$

I.e., there is a consensus among all partners that the value of piece j is *nearly-exactly* w_j , where the difference is less than ε .^{[1]:127}

2. Perfect Division

A **perfect division** is a division in which a resource is divided among n partners with subjective valuations, giving each partner exactly $1/n$ of the resource according to the valuations of *all* partners. It is a special case of exact division in which all weights are $1/n$.

3. Exact Division with Arbitrary Number of Cuts

3.1. Piecewise-Homogeneous Cake, Many Partners, Any Weights

A cake is called *piecewise-homogeneous* if it can be divided to R regions, such that all partners agree that the value density in each region is uniform. For example, consider a circular cake in which each of its 4 quarters has a different topping. The partners may value each of the toppings differently, but do not distinguish between different pieces having the same topping. This means that the value of each piece to each partner only depends on the *amount* they get from each region.

Saying that the cake is piecewise-homogeneous is equivalent to saying that the valuations of the partners are *piecewise-constant*: each piece of the cake is homogeneous if and only if it is the intersection of n pieces of the n partners.

whenever the cake is piecewise-homogeneous (and the valuations are piecewise-constant), a consensus division can be achieved in the following way:

- Divide each region to k sub-regions, such that sub-region j contains exactly w_j of the regions.
- Let piece j be the union of the j -th sub-regions in all R regions. This defines a consensus division with the given weights.

This algorithm can be generalized to slightly more general families of value measures, such as piecewise linear.^[2]

The number of required cuts is kR , where R is the number of regions.

3.2. General Cake, Many Partners, Any Weights

When the value measures are countably-additive and non-atomic, a consensus partition exists for every set of weights whose sum is 1. This is a corollary of the Dubins–Spanier convexity theorem.

Woodall^[3] showed that it is possible to construct such a division of an interval cake as a countable union of intervals.

INTUITION: Consider the division procedure for piecewise-homogeneous cakes described above. In general, the cake is not piecewise-homogeneous. However, because the value measures are continuous, it is possible to divide the cake to smaller and smaller regions such that the regions become more and more homogeneous. When $R \rightarrow \infty$, this process converges to a consensus division.

Fremlin showed that it is possible to construct such a division as a *finite* union of intervals.

Stromquist and Woodall^[4] showed that it is possible with the *minimal* number of intervals; see Stromquist–Woodall theorem.

4. Exact Division with a Minimal Number of Cuts

Suppose the cake is an interval made of n districts (sub-intervals), and each of the n partners values only a single district. Then, a consensus division of the cake to k subsets requires $n \cdot (k - 1)$ cuts, since each of the districts must be cut to k pieces which are equal in the eyes of the partner that values this district. Hence, it is an interesting question whether it is **always** possible to attain a consensus division with this exact number of cuts.

4.1. Interval Cake, Two Partners, Many Subsets, Any Weights

Two partners can achieve a consensus division using Austin moving-knife procedure.

The simplest case is when the weights are $1/2$, i.e. they want to cut a piece that both of them agree to be half the cake value. This is done as follows. One partner moves two knives over the cake from left to right, always keeping the value between the knives as exactly $1/2$. It is possible to prove (by the intermediate value theorem) that at some point, the value of the piece between the knives to the other partner will also be exactly $1/2$. The other partner calls "stop!" at that point and the piece is cut.

The same protocol can be used to cut a piece that both player agree that its value is exactly $1/n$.

By combining several such pieces, it is possible to achieve a consensus division with any ratios that are rational numbers. But this may require a large number of cuts.

A better way to achieve a consensus division is to identify the two endpoints of the cake and treat it like a circle. I.e, when the right knife gets to the right side, it immediately goes to the left side, and the piece-between-the-knives is now actually the union of the piece to the right of the right knife and the piece to the left of the left knife. This way, it is possible to find a consensus division for every $p \in [0, 1]$. One partner moves the knives cyclically around the cake, always keeping the value between them at exactly p . It is possible to prove that at some point, the value of the piece between the knives to the other partner will also be exactly p .^[5] The other partner calls "stop!" at that point and the piece is cut. This requires only two cuts.

By repeatedly applying the above procedure, it is possible to achieve a consensus division to two partners and any number of subsets. The number of cuts is $2(k - 1)$, where k is the number of subsets.

As of 2015, there is no known generalization of this moving-knife procedure to more than 2 partners.^[6]

4.2. Interval Cake, Many Partners, Two Subsets, Equal Weights

Suppose the cake is an interval of value 1. It should be divided into $k = 2$ subsets, each of which has a value of exactly $1/2$ to all n partners. We want to use the minimal number of cuts, which is $n(k - 1) = n$.

A division like this always exists.^[7] This is a direct corollary of the Hobby–Rice theorem. This can also be proved based on the Borsuk-Ulam theorem:^[8]

- Every partition of an interval using n cuts can be represented as a vector of length $n + 1$, in which the elements are the lengths of the sub-intervals.
- Every element of the vector can be either positive (if it belongs to piece #1) or negative (if it belongs to piece #2).
- The set of all partitions is the sphere S^n .
- Define a $V : S^n \rightarrow \mathbb{R}^n$ in the following way: for every partition x , $V(x)$ is a vector whose i -th element is the value of piece #1 in that partition according to partner i , minus $1/2$.
- The function V is continuous. Moreover, for all x , $V(x) + V(-x) = 0$.
- Hence, by the Borsuk-Ulam theorem, there exists an x such that $V(x) = 0$. In that partition, all partners value piece #1 (and piece #2) as exactly $1/2$.

A consensus division to two subsets can be found based on Tucker's lemma, which is the discrete version of Borsuk-Ulam theorem.^[9]

Although the partners' preferences are modeled with measures, the proofs do not require the value measures to be additive over subsets. The value measures may as well be continuous set functions defined on the Borel sigma-algebra and satisfying all the properties of measures except countable additivity. Thus it is not required that partners' valuations over subsets of the cake be additively separable.^[9]

4.3. Interval Cake, Many Partners, Many Subsets, Equal Weights

The existence theorem of the previous subsection can be generalized from $k = 2$ pieces to an arbitrary number of pieces. This was proved by Noga Alon in his 1987 paper about the necklace splitting problem.

There are n different measures on the interval, all absolutely continuous with respect to length. The measure of the entire necklace, according to measure i , is $k \cdot a_i$. Then it is possible to partition the interval to k parts (not necessarily contiguous), such that the measure of each part, according to measure i , is exactly a_i . At most $(k - 1)n$ cuts are needed, and this is optimal.

4.4. Interval Cake, Many Partners, Two Subsets, Arbitrary Weights

The existence theorem of the previous subsection is generalized to arbitrary weights by the Stromquist–Woodall theorem.

4.5. Multi-Dimensional Cake, Many Partners, Many Subsets, Equal Weights

The Stone–Tukey theorem states that given n measurable "objects" in n -dimensional space, it is possible to divide all of them in half (with respect to their measure, i.e. volume) with a single $(n - 1)$ -dimensional hyperplane.

Stated differently: if the cake is the space \mathbb{R}^n , and the value measures of the partners are finite and vanish on any $n - 1$ dimensional hyperplane, then there is a half-space whose value is exactly $1/2$ to each partner. Hence there exists a consensus division using a *single* cut.

The original version of this theorem works only if the number of dimensions of the cake is equal to the number of partners. E.g, it is not possible to use this theorem to divide a 3-dimensional sandwich to 4 or more partners.

However, there are generalizations that enable such a division. They do not use a hyperplane knife but rather a more complicated polynomial surface.^[10]

5. Near-exact Division Procedures

5.1. Crumb-and-Pack Procedure

For any given $\varepsilon > 0$, one can give each partner a piece such that all partners believe that the values they have differ by less than ε , i.e., for every i and every j :^{[1]:127}

$$|V_i(X_j) - w_j| < \varepsilon$$

The near-exact division procedure has two steps: *crumbing* and *packing*.

Crumbing step: the goal is to cut the cake to tiny bits ("crumbs") such that each partner assigns a sufficiently small value to each crumb. This is done in the following way. Let k be a certain constant. Ask partner #1 cut the cake to k pieces that he values as $1/k$. Ask partner #2 to trim pieces as needed (using at most $k-1$ cuts) such that each piece has a value of at most $1/k$. These new pieces of course still have a value of at most $1/k$ for partner #1. Continue with partners #3, #4, ..., # n . Finally all n partners value each resulting crumb as at most $1/k$.

Packing step: the goal here is to partition the crumbs to n subsets, such that the sum of values in each subset j is near w_j . Here is an intuitive explanation of the packing step for two partners (Alice and George) when the weights are $1/2$.^[1]68–

1. Get an empty bowl.
2. Insert into the bowl one of the crumbs.
3. If the value in the bowl becomes more than $1/2$ to either partner, give the bowl to that partner and give the other crumbs to the other partner.
4. Otherwise (the value in the bowl is less than $1/2$ to both partners), if the value in the bowl is larger for Alice than for George, then find a crumb whose value for George is more than its value for Alice (such a crumb must exist because the sum of values of all crumbs is 1 both for Alice and for George). Add this crumb to the bowl and return to step 2.

It is possible to prove by induction, that the difference in the valuation of the bowl between Alice and George is always at most $1/k$. Hence, when one of the partners receives the bowl, its value for both partners is between $1/2 - 1/k$ and $1/2 + 1/k$.

Formally, each piece can be represented as a vector of values, one per partner. The length of each vector is bounded, i.e. for each such vector v : $\|v\| \leq \sqrt{n}/k$. Our goal is to create, for each partner j , a vector all whose elements are near w_j . To do this, we have to divide the vectors to subsets, such that the sum of vectors in each subset j is sufficiently close to a vector all whose elements are w_j . This is possible thanks to a theorem by V.Bergström,^{[1][11]:126–128}

The Crumb-and-Pack procedure is a subroutine in the Robertson-Webb protocol. The latter protocol generates a division which is both near-exact and envy-free cake-cutting.

A different explanation of the crumb-and-pack procedure is provided by Brams and Taylor.^[12]

6. Truthful Mechanisms

Any algorithm for consensus division relies on the value measures reported by the partners. If the partners know how the algorithm works, they may have an incentive to lie about their value measures in order to receive more than their weight. In order to prevent this, a truthful mechanism should be used.^{[2][13]}

The simplest truthful division mechanism is: select a single partner at random (with probabilities determined by the weights) and give him the entire cake. This mechanism is trivially truthful because it asks no questions. Moreover, it is consensus in expectation: the expected value of each partner is exactly its weight, and this is true according to all value measures. However, the resulting division is of course not a consensus division.

A better truthful mechanism, which works for the case in which all weights are $1/n$, can be built given any existing algorithm (or oracle) for finding a consensus division:

1. Ask each partner to report his value measure.
2. Use the existing algorithm/oracle to generate a partition in which all n pieces are exactly $1/n$ according to the value functions reported by the partners.
3. Perform a random permutation on the consensus partition and give each partner one of the pieces.

Here, the expected value of each partner is still $1/n$ regardless of the reported value function, so the mechanism is still truthful – no partner can gain anything from lying. Moreover, a truthful partner is guaranteed a value of exactly $1/n$ with probability 1 (not only in expectation). Hence the partners have an incentive to reveal their true value functions.

7. Impossibility

It is impossible to achieve an exact division with a finite number of queries, even if there are only 2 partners and the weights are exactly $1/2$.^{[1]:103–104} This means that the best we can achieve using a discrete algorithm is a near-exact division.

Proof: When the protocol is at step k , it has a collection of at most k pieces. To provide an exact division, the protocol must find an *exact subset* – a subset of the pieces which both partners value as exactly $1/2$. We are going to prove that, for every k , there are situations in which at step k there is no exact subset, and hence the protocol might have to continue endlessly.

Initially, there is only one piece which both partners value as 1, so there is obviously no exact subset. After one step, at most one partner (say, Alice) has had an option to cut the cake. Even if Alice cuts the cake to two pieces that are equal in her opinion, they may be different in George's opinion, so again there is no exact subset.

Suppose now that we are at step k and there are k pieces. Without loss of generality, we may assume that each piece has a non-zero value to both partners. This is because, if Alice (for example) cuts a piece which she values as 0, it is possible that George also values the same piece as 0, so we can discard this piece and continue with the other pieces.

The total number of different subsets now is 2^k , and by the induction assumption none of them is exact. At step k , the protocol can ask either Alice or George to cut a certain piece to two pieces. Suppose w.l.o.g. that the cutter is George and that he cuts piece X to two sub-pieces: X_1 and X_2 . Now, the total number of subsets is 2^{k+1} : half of them already existed and by assumption they are not exact, so the protocol's only chance of finding an exact subset is to look at the new subsets. Each new subset is made of an old subset in which the piece X has been replaced with either X_1 or X_2 . Since George is the cutter, he can cut in a way which makes one of these subsets an exact subset for him (e.g. if a certain subset containing piece X had a value of $3/4$, George can cut X such that X_1 has a value of $1/4$ in his opinion, so that the new subset has a value of exactly $1/2$). But, George does not know Alice's valuation and cannot take it into account when cutting. Therefore, there is an uncountable infinity of different values that the pieces X_1 and X_2 can have for Alice. Since the number of new subsets is finite, there is an infinite number of cases in which no new subset has a value of $1/2$ for Alice, hence no new subset is exact.

8. Comparison with Other Criteria

An exact division with equal weights ($1/n$) is, in particular, also proportional, envy-free and equitable.

However, it is not necessarily Pareto efficient, since in many cases it is possible to take advantage of the subjective valuations and divide the resources such that all partners receive more than their fair share of $1/n$.

Exact divisions are much easier if the participants cooperate in establishing entitlements rather than competing as in fair division. Some authors refer to this as **consensus division** or **consensus halving**.^[14]

9. Summary Table

Name	Type	Cake	Valuations ^[15]	#partners (n)	#subsets (k)	#cuts	weights
Austin	Moving-knife procedure	Interval	Con	2	Many	$2(k-1)$ (optimal)	Any
Piecewise-homogeneous	Discrete procedure	Piecewise-homogeneous	Con+Add+Pwl	Many	Many	Num. of districts	Any
Dubins–Spanier	Existence proof	Any	Con+Add	Many	Many	Unbounded	Any
Consensus-halving	Infinite procedure	Interval	Con	Many	2	n (optimal)	Equal
Necklace-splitting	Existence proof	Interval	Con(+Add?)	Many	Many	$n(k-1)$ (optimal)	Equal
Stromquist-Woodall	Existence proof	Circle	Con+Add	Many	2	$2n-2$ (optimal for some weights)	Any

Stone–Tukey	Existence proof	n-dimensional	Con(+Add?)	n	2	1 half-plane	Equal
Crumb-and-pack	Near-exact procedure	Any	Con+Add	Many	Many	Unbounded	Any

References

1. Template:Cite Robertson Webb 1998
2. Chen, Yiling; Lai, John K.; Parkes, David C.; Procaccia, Ariel D. (2013). "Truth, justice, and cake cutting". *Games and Economic Behavior* 77 (1): 284–297. doi:10.1016/j.geb.2012.10.009. <http://nrs.harvard.edu/urn-3:HUL.InstRepos:8896229>.
3. Woodall, D.R (1980). "Dividing a cake fairly". *Journal of Mathematical Analysis and Applications* 78: 233–247. doi:10.1016/0022-247x(80)90225-5. <https://dx.doi.org/10.1016%2F0022-247x%2880%2990225-5>
4. Stromquist, Walter; Woodall, D.R (1985). "Sets on which several measures agree". *Journal of Mathematical Analysis and Applications* 108: 241–248. doi:10.1016/0022-247x(85)90021-6. <https://dx.doi.org/10.1016%2F0022-247x%2885%2990021-6>
5. Fischer, Daniel. "Consensus division of a cake to two people in arbitrary ratios". Math.SE. <https://math.stackexchange.com/q/1333054>. Retrieved 23 June 2015.
6. There is a generalization which gives each of n partners, a piece worth exactly $1/n$ for him. But this is not a consensus division, because the partners may not agree on the value of the other pieces besides the piece allocated to them. See Austin moving-knife procedures#Many partners.
7. Goldberg, Charles H.; West, Douglas B. (1985). "Bisection of Circle Colorings". *SIAM Journal on Algebraic and Discrete Methods* 6: 93–106. doi:10.1137/0606010. <https://dx.doi.org/10.1137%2F0606010>
8. Alon, Noga; West, Douglas B. (1986). "The Borsuk-Ulam theorem and bisection of necklaces". *Proceedings of the American Mathematical Society* 98 (4): 623. doi:10.1090/s0002-9939-1986-0861764-9. <https://dx.doi.org/10.1090%2Fs0002-9939-1986-0861764-9>
9. Simmons, Forest W.; Su, Francis Edward (2003). "Consensus-halving via theorems of Borsuk-Ulam and Tucker". *Mathematical Social Sciences* 45: 15–25. doi:10.1016/S0165-4896(02)00087-2. <https://dx.doi.org/10.1016%2FS0165-4896%2802%2900087-2>
10. B. Grünbaum (1960). "Partitions of mass–distributions and convex bodies by hyperplanes". *Pacific J. Math.* 10 (4): 1257–1261. doi:10.2140/pjm.1960.10.1257. <https://dx.doi.org/10.2140%2Fpjm.1960.10.1257>
11. V. Bergström (1930). "Zwei Sätze über ebene Vectorpolygone". *Hamburgische Abhandlungen* 8: 205–219.
12. Brams, Steven J.; Taylor, Alan D. (1996). *Fair Division*. pp. 131–133. ISBN 978-0-521-55644-6.
13. Mossel, Elchanan; Tamuz, Omer (2010). Truthful Fair Division. *Lecture Notes in Computer Science*. 6386. pp. 288–299. doi:10.1007/978-3-642-16170-4_25. ISBN 978-3-642-16169-8. https://dx.doi.org/10.1007%2F978-3-642-16170-4_25
14. de Longueville, Mark; Živaljević, Rade T. (2008). "Splitting multidimensional necklaces". *Advances in Mathematics* 218 (3): 926–939. doi:10.1016/j.aim.2008.02.003. <https://dx.doi.org/10.1016%2Fj.aim.2008.02.003>
15. Pre-requisites on the value functions of the partners. Less pre-requisites mean that the result is more general. Con=Continuous is the most general; Con+Add=Additive is less general; Con+Add+Pwl=Piecewise-linear is the least general.

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