Comparisons of Generalized Schwarzschild Spacetimes in the Strong-Field Regime

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Some of the theorems about the line element of the blackhole spacetimes are reviewed. The foliation of the spacetimes is recalled. The comparison of ENBI gravity and that of massive gravity is achieved. The strong-field regime of blackhole objects is investigated.

Generalized Schwarzschild spacetimes | Blackhole masses | spacetime foliations

1. Introduction

Different blackhole solutions are compared. For this aim, the Theorems about blackhole spacetimes which allow for a generalization of the Schwarzschild black hole as are recalled. The structure of the spacetimes which allow the presence of Killing tensors are reviewed- as a result, the theoretical foundations of the comparison is set.

The comparison if the line elements is therefore possible, in order for the different addends to be identified. From [1], The Hawking-Page transition does not take place for the adS blackhole objects with Ricci-flat horizons and for the adS blackhole objects with Ricci hyperbolic horizons; this consideration motivates some of the studies here presented. The study of the experimental evidences about blackhole masses is further reviewed.

2. Introductory Material

2.1. Theorems about Blackhole Spacetimes

From [2], the blackhole object of generalised Schwarzschild gtt component of the line element as

$$ds^2 = -dt^2 V^2(t) + g_{ij} dx^i dx^j \qquad (1)$$

on a slice Σ such that it is a $t = 0$ slice, with $V > 0$ on the slice Σ. The fundamental hypothesis is requested, that there exists a compact set $k \subset \Sigma$ such that $k \sim \Sigma$ diffeomeorphic to $R^3 \sim {}^{-}B_1(0)$, where ${}^{-}B_1(0)$ is a closed spherical neighborhood centered at the origin: on Σ, the space components of the metric tensors are split as $g_{ij} = \delta_{ij} + h_{ij}$. Ibidem, the functional dependence of V on the coordinates is proven as

$$V = 1 - \frac{\tilde{m}}{\mid Y \mid} + v \qquad (2)$$

where $\mid y \mid^2 = \sum_{i=1}^{i=3} \mid y_i \mid^2 \to \infty$

As a result, the added terms v in Eq. (2) isproven to be requested to be

$$v = \mathcal{O}(\mid Y \mid^{-2}). \qquad (3)$$

The result of [3] is recalled, according to which the $_{ij}$ components of the metric tensor are

$$\gamma_{ij} = 1 + \frac{2c}{\mid Y \mid}\delta_{ij} + a_{ij}, \qquad (4)$$

where a_{ij} is requested to be

$$a_{ij} = \mathcal{O}(\mid Y \mid^{-2}). \qquad (5)$$

In the present analysis, the Y value is chosen as the radial coordinate R, and the absolute-value specification becomes redundant.

Such blackhole objects are ibidem proven to be unique, i.e. the existence of 'multiple blackholes in an asymptotically-Euclidean static vacuum spacetimes is discarded. The uniqueness of static charged blackhole spacetimes was studied in [4]; there the work of [2] was specified for the space components of the metric tensor from Eq. (1) to be as

$$g_{ij} = \left(1 - \frac{2m}{r}\right)\delta_{ij} + h_{ij} \qquad (6)$$

with the components of Eq. (2) unchanged. The hypotheses of [4] were refined from [5].

2.2. Non-Trivial Killing Tensors of Blackhole Spacetimes

In [6], non-trivial Killing tensors are reviewed, and their use in the definition of generalized photon surfaces is given.

In [7], a timelike hypersurface is newly defined in a way such that any worldline of a particle of mass m, electric charge q, electromagnetic potential Aμ and fixed total energy E which initially touches it is going to stay on the defied hypersurface forever.

Conditions for ’margianlly-stable’ orbits are ibidem found.

Ibidem, the kinetic energy Ekin and the potential energy Epot are introduced such that

$$E_{pot} \equiv -qk_\alpha A^\alpha, \qquad E_{kin} \equiv E - E_{pot} \equiv -k_\alpha v^\alpha(E), \quad v_\alpha v^\alpha = -m^2. \tag{7a}$$

The considered spacetimes are from [8] timelike outside the ergosphere as

$$\pounds_k g_{\mu\nu} = 0, \tag{8a}$$
$$\pounds_k A_\alpha = 0. \tag{8b}$$

The paradigm is apt for metrics obeying the Birkhoff Theorem. The total energy of a particle E considered is

$$E \equiv -k^\alpha(\dot{\gamma}^\alpha + qA^\alpha) \tag{9}$$

The complete Killing maps of the normal component and of the tangent component was taken from [6][9].

2.3. The Colombeau Geometry for the Neighbourhood of the (Schwarzschild) Fictitious Singularity

In [10], the Colobeau theory is exposed as far as the definition of the distributional spacetimes, whose results, as well as those of the related items of bibliography, will be made use of in SCRIVI.

From [11], the physical interpretation of the ’distributional Schwarzschild geometry’ is explained.

From [10], in Remark 1.1.4, G(M′) is defined as the algebra of the Colombeau generalized functions M′ ⊂ M. ˜ R is the set of the Colombeau generalized numbers [12][13][14].

Let (gϙ)ϙ the Colombeau generalized metric tensor on M.

Rμν M′ is the generalized Ricci tensor on M; furthermore, as from [15][16],

Rμν M′(p) is the generalized Ricci tensor on M of the metric (gϙ(p))ϙ | M′.

There is a degenerate singular metric tensor from the Ricci tensor as

$$R_{\mu\nu\ \mathcal{M}'}(p) \in \mathcal{G}(\mathcal{M}')\backslash c^\infty(\mathcal{M}'), \tag{10}$$

i.e.

$$\forall p \in \mathcal{M}' : R_{\mu\nu\ \mathcal{M}'}(p) \in \tilde{\mathbb{R}}\backslash\mathbb{R}.$$

From Definition 1.1 ibidem, (G(M′)) is the algebra of Colombeau generalized functions on M′ ⊂ M.

The Colombeau distributional scalar curvature $R_M(p)$ is defined as

$$R_{\mathcal{M}}(p) = tr[(R_{\mu\nu\ \epsilon\ \mathcal{M}}(P))_{\epsilon}] \qquad (11)$$

under the hypothesis

$$R_{\mathcal{M}}(p) \subset \mathcal{G}(\mathcal{M}')\backslash c^{\infty}(\mathcal{M}'). \qquad (12)$$

The 'distributional spacetime' of metric tensor $(g_{\mu\nu\ \varrho}(p))_{\varrho}$ exhibits a singularity on a smooth compact submanifold MC ⊂ M iff RMc (p) ∈ G(MC\c∞(MC).

Furthermore, the distributional spacetime (gϱ(p))ϱ has a singularity with compact support iff RM_c (p) ∈ D(R^3).

2.4. The Generalized Photon Surfaces from the Killing Tensors

In [6], the definition of generalized photon surfaces of blackhole spacetimes is derived from the notion of non-trivial Killing tensors.

Massive-particle surfaces (MPS's) are introduced in [7] as generalizations of the concepts developped in [17].

From [8], Conformal Killing Vectors (CKV's) and Conformal Killing Tensors (CKT's) are outlined for arbitrary null geodesics with affine parametrization.
Reducible CKT's are recalled to be constructed from symmetrization procedures from [18][19].

The paradigm schematized in [7], the results of [17] are extended to the instance of massive charged particles.

In [17], generalization of the concept of photon sphere of a blackhole object is extended from that of the Schwarzschild spacetimes.

Photon surfaces are defined ibidem after energy conditions, i.e. they are defined as the surfaces ($M,g^{\mu\nu}$) as 'immersed', 'no-where-spacelike' hypersurface S of
(M,gμν) such that, for all p point of S and for all k^μ null vector in the tan gent space of S, TpS, there exists a null geodesics γ such that (−ξ,ξ) → M of ($M,g^{\mu\nu}$) in the way defined as ˙ γ(0) = k^μ, | γ |∈ S.

3. The EN-BI Gravitational Model

As recalled from [20] from [21], the action I is taken as

$$\mathcal{I} = -\frac{1}{16\pi}\int d^x\sqrt{-g}\left[R - 2\lambda + \mathcal{L}(\mathcal{F}) + m^2\sum_{i=1}^{i=4} c_i\mathcal{U}_i(g,f)\right] \qquad (13)$$

where R is the Ricci curvature, the negative cosmological-constant term λ is written as

$$\lambda \equiv \frac{(d-1)(d-2)}{2l^2} \qquad (14)$$

f is a fixed symmetric tensor, the Faraday tensor Fμν ≡ ∂μAν − ∂νAμ is satu rated as F as

$$\mathcal{F} = F^{\mu\nu}F_{\mu\nu}, \qquad (15)$$

the polynomials $U_i(g,f)$ are polynomials of the eigenvalues of of the tensor $K^{\mu}\nu$. The BI electrodynamics is defined after the Lagrangean density L(F) as

$$\mathcal{L}(\mathcal{F}) \equiv 4\beta^2\left[1 - \sqrt{1 + \frac{\mathcal{F}}{2\beta^2}}\right]. \qquad (16)$$

The fields equations are obtained as follows. Variation with respect to the metric tensor $g_{\mu\nu}$ lead to the EN-BIFE’s

$$G_{\mu\nu} + \lambda G_{\mu\nu} - \frac{1}{2}\mathcal{L}(\mathcal{F}) - 2F_{\mu\sigma}F^{\sigma}{}_{\nu} + m_0\chi_{\mu\nu} = 0, \qquad (17)$$

with the tensor $\chi_{\mu\nu}$ as from [20]. The symmetric tensor $f_{\mu\nu}$ is here chosen as

$$f_{\mu\nu} \equiv (0, 0, c^2 h_{ij}), \qquad (18)$$

being h_{ij} the pertaining line element of a chosen Euclidean space, and c and arbitrary constant. Variation with respect to the Faraday tensor $F_{\mu\nu}$ leads to the equality

$$\partial_\mu \frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1 + \frac{\mathcal{F}}{2\beta^2}}} \qquad (19)$$

In [20], the spherically-symmetric non-rotating blackhole spacetime metric is chosen in an arbitrary number of dimensions as

$$ds^2 = -f(r)dt^1 + \frac{1}{f(r)}dr^2 + r^2 h_{ij} dx^i dx_j \qquad (20)$$

where the symmetric tensor $f_{\mu\nu}$ from Eq. (18) has been specified to the 4 dimensional case, and for a Schwarzschild solid angle.

3.1. Some Limiting Processes in EN-BI Gravitational Model in the 4-Dimensional Case

The metric Eq. (20) is here specified in 4 spacetime dimensions for a spherically symmetric non-rotating blackhole spacetime endowed with a Schwarzschild solid angle element. The line element Eq. (20) therefore rewrites

$$ds^2 = -f(r)dt^1 + \frac{1}{f(r)}dr^2 + r^2(sin\theta)^2 d\phi^2 + r^2 d\theta^2 \qquad (21)$$

The g_{tt} function Eq. (20) is developped to

$$f(r) = k + \frac{m_0}{r^2} + \left(\frac{4\beta^2 - 2\lambda}{6}\right) r^2 - \frac{2\beta^2}{3}\sqrt{1+\mathcal{G}} + \frac{8}{3}\frac{\mathcal{H}}{r^2} + m^2\left(\frac{cc_1}{2}r + c^2 c_2 + \frac{c^3 c_3}{r}\right) : \qquad (22)$$

the constant m0 is an integration constant which is related to the blackhole mass (i.e. according to the chosen limit to the Nariai spacetimes), and the constant G is defined as related to the electric charge q as

$$\mathcal{G} = \frac{2q}{\beta^2 r^4}. \qquad (23)$$

In the present Section, a vanishing electric charge is assumed at the moment. The g_{tt} component of the line element Eq. (20) is therefore reconducted to the following expression

$$f(r) \equiv k + m^2 + c^2 c_2 + m^2 \frac{c^3 c_3}{r} + \frac{m_0}{r^2} + m^2 \frac{cc_1}{2} r + \frac{4\beta^2 - 2\lambda}{6} r^2 \qquad (24)$$

4. Conformal Gravity Theories

In [22], Weyl theories of gravity [23][24][25][26][27][28] are compared as far as the components of the metric tensor are concerned with massive theories of gravities.

The chosen line element is one of static spherically-symmetric spacetimes with line element

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij} dx_i dx^j, \tag{25}$$

with the component of the metric tensor h_{ij} as

$$h_{ij} = d\theta^2 + d\phi^2, \quad k = 0, \tag{26a}$$
$$h_{ij} = d\theta^2 + (sin\theta)^2 d\phi^2, \quad k = 1, \tag{26b}$$
$$h_{ij} = d\theta^2 + (sinh\theta)^2 d\phi^2, \quad k = -1. \tag{26c}$$

In [29] and in [30], solutions of the EFE's for the conformal transformation $g\mu\nu \rightarrow \Omega^2(x)g^{\mu\nu}$ with Ω a non-singular function of the coordinates is reconducted to the expression

$$f(r) = s_0 + rs_1 + \frac{s_2}{r} - \frac{\Lambda}{3}r^2 \tag{27}$$

with the constants s_i and Λ. From [30], the Schwarzschild limit is recovered in the instance

$$s_0^2 = 3s_1 s_2 + k^2. \tag{28}$$

From [31] and from [20], the comparisons are found

$$f(r) = k - \frac{r_s}{2r} - \frac{\Lambda}{3}r^2 + m^2\left(\frac{cc_2}{2}r + \frac{c^2 c_2^2}{}\right) \tag{29}$$

The Schwarzschild radius is found to define the parameter space of the model as

$$M = -\frac{2}{3}\frac{cc_2}{c_1}(2k + m^2 c^2 c_2) \tag{30}$$

5. Comparisons of Blackhole Solutions

It is possible to confront EN-BI blackhole solutions, conformal-gravity blackhole solutions, and generalized Schwarzschild spacetimes blackhole solutions. The method of [22] is here newly developped. From Eq. 6 of [22]

$$q(r) = s_0 + \frac{s_2}{r} + s_1 r + s_3 r^2 \tag{31}$$

Comparison of blackhole solution of conformal gravity and of massive gravity is based on the equality

$$\tilde{q}(r) = k + m^2\mathcal{C}^2 c_2 - \frac{M}{r} + m^2\frac{\mathcal{C}c_1}{2}r + s_3 r^2 \tag{32}$$

5.1. Comparison with 4-dim Weyl Gravity

Compasion of the 4-dimensional Weyl gravity solutions and the chosen of the EN-BI gravitational blackhole spaceitmes are newly found after imposing

$$s_0 = k + m^2\mathcal{C}^2 c_2 \equiv 1, \tag{33a}$$
$$s_1 = \frac{m^2\mathcal{C}c_1}{2}, \tag{33b}$$
$$s_2 = -M. \tag{33c}$$

The request is here posed

$$s_0 = s_0^2 = 1 \tag{34}$$

i.e.

$$s_0 = 1 = k + m^2\mathcal{C}^2 c_2 = k^2 - 3s_1 M = k^2 - \frac{3}{2}Mm^2\mathcal{C}c_1 \tag{35}$$

The solution is looked for for M. To this aim, one remarks that the limit m → ∞ is well-posed in the case of asymptotical flatness and in the case k = 1. In the case of asymptotical flatness, the comparison is possible with the Kottler Schwarzschild-Kiselev spacetimes, as equality Eq. (35) is assured to be well posed because the Kottler-Schwarzschild Kiselev spacetimes obey the Birkhoff Theorem, as from [32].

6. The Preparation of Particle Fields Close to the Blackhole Fictitious Singularity in the Strong Field Regime

In [33], the preparation of particle fields close to the surface of a blackhole object is discussed.

The n-dimensional spacetimes with metric $g_{\alpha\beta}$ are considered, with a time-like killing vector k[1]*alpha* i considered, and electro-magnetic of covector A_α with the same symmetries as

$$\pounds_k A_\alpha = 0. \tag{36}$$

For a worldline γ, the particle 4-position γ^α and the particle 4-velocity (rvelocity) $\dot{\gamma}^\alpha$ are looked for. For these purposes, the total energy E is calculated, as well as the kinetic energy Ekin and the potential energy E_{pot}: this way, the configuration of a particle is

$$E_{kin} = -k_\alpha \dot{\gamma}^\alpha, \quad (37a)$$
$$E_{pot} = -q k_\alpha A^\alpha. \quad (37b)$$

Massive-particle surfaces (MPS') are described in [7] and in [34], while hidden symmetries are investigated in [35] and in [36].

MPS' are described from [7] as (n − 1)-dimensional timelike hypersurfaces'exhibiting a space outer normal n^α: any particle with fixed energy E, i.e. such

that the MP starts in S with initial velocity and remains in S always. It is translated as

$$h_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta, \quad (38a)$$
$$\varpi_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta \nabla_\mu n_\nu. \quad (38b)$$

Let κ be the 'projection of the time-like Killing vector onto S:

$$\kappa = h^\alpha{}_\beta k^\beta : \quad (39)$$

here, the Killing vector is hypothesized to be tangent to S as

$$\kappa^\alpha = k^\alpha, \quad (40)$$

i.e. the MP is defined to be 'static'.

The connections between the surfaces and the hidden symmetries allows one to reconduct the shadow of the blackhole [37], where also interactions with plasma are considered.

6.1. The 4-Dimensional Models

A general static axisymmetric 4-dimensional spacetime is considered with line element

$$ds^2 = -\alpha dt^2 + \varphi dr^2 + \beta d\phi^2 + \gamma d\theta^2 \quad (41)$$

where the components of the metric tensor Eq. (41) are functions of the 4 dimensional variables r and θ.The time-like Killing vector $k^{\alpha}\partial_{\alpha} = \partial_t$ implies

$$k^2 \equiv -\alpha \tag{42}$$

as
$$\pounds_{\partial t} A_{\alpha} = 0. \tag{43}$$

A MPS ansatz is taken from [33] that the particle surface is as

$$r = const, \tag{44}$$

which is achieved with

$$k = -\frac{1}{\sqrt{\varphi}}\partial_r ln\frac{\beta}{\alpha} \tag{45}$$

The resulting energy E is found as a function of the mass of the particle μ as

$$E_{\pm} = \pm\mu\sqrt{\alpha\frac{\partial\beta}{\partial_r ln(\beta\alpha)} + \frac{F_{rt}^2}{(\partial_r ln(\beta/\alpha))^2}\frac{q^2}{\mu^2}} - \frac{F_{rt}}{\partial_r ln(\beta/\alpha)}q - qA. \tag{46}$$

At q =0, the conditions imply

$$\partial_r\left(\frac{\beta}{\gamma}\right) = 0, \tag{47a}$$

$$E_{\pm} = \mu^2\alpha\frac{\partial_r ln\beta}{\partial_r ln(\beta\gamma)}. \tag{47b}$$

The photon sphere is located at q=μ=0. The condition K=0 implies the total-umbilicity

$$\partial_r ln\alpha = \partial_r ln\beta = \partial_r ln\gamma \tag{48}$$

The Schwarzs child black hole space time is defined as

$$\alpha = \varphi^{-1}, \tag{49a}$$
$$\beta = r^2, \tag{49b}$$
$$\gamma = r^2(sin\theta)^2. \tag{49c}$$

Eq.'s(47a) imply for the energy E the definition

$$E_{\pm} = \pm\mu^2 \frac{2f^2}{2f - rf'} \tag{50}$$

with inner most stable orbit as

$$\frac{\partial E}{\partial r} = 0. \tag{51}$$

The equation of the photon surface is obtained from Eq. (50) as

$$2f - rf' = 0. \tag{52}$$

Uniqueness theorems are presented in [38] and in [35].

6.2. Reissner-Nordstroem Spacetimes

Reissner-Nordstroem spacetimes are qualified after the components of the metric tensor

$$f(r) = 1 - \frac{r_s}{r} + \frac{r_Q}{r^2}. \tag{53}$$

The energy $E_{\pm}$ is calculated as

$$E_{\pm} = \mu^2 \frac{(r(r - r_S) + r_Q^2)^2}{r^2 \left[r\left(r - \frac{3}{2} r_S\right) + 2r_Q^2 \right]}. \tag{54}$$

The radius of the photon surface r_{ps} is calculated as

$$r_{ps} = \frac{1}{2}\left(\frac{3}{2} r_S + \sqrt{\frac{9}{4} r_S^2 - 8 r_Q^2} \right) \tag{55}$$

7. Cosmological Implementation

The cosmological implementation of the models is here presented from [39] also as far as the thermal history of the universe is concerned.

In [39], analysis of a'narrow atmospheric layer' in a suitably-chosen spherical neighborhood of the Astrophysical blackhole is considered. Applications can be considered to new explanation of the speed of growth of supermassive blackholes and related objects in the Early Universe [40][41][42][43]. Furthermore, an explanation of the low mass gap [44][45][46][47] in newly obtained.

7.1. Thefictitious Singularities of the Schwarzschild Metric

A regularized Schwarzschild radial variable is taken from [10][48] as r_ϱ as

$$r \to r_\epsilon = r_S + \sqrt{(r-r_S)^2+\epsilon^2} \tag{56}$$

at which the Ricci scalar is explained as one with a singular behaviour as

$$lim_{\epsilon\to 0} \sim \delta(r-r_S). \tag{57}$$

The non-smooth Colombeau regularization can also be chosen, i.e. as from [11] and from[49].

A thin spherical neighbourhood is chosen as

$$r_S < r < \frac{3}{2} r_S. \tag{58}$$

Isotropic coordinates are discussed in [50].

In isotropic coordinates, the line element is rewritten as

$$ds^2 = N(\rho)dt^2 - \Psi^4(\rho)\left[d\rho^2 + \rho^2 d\theta^2 + \rho^2 (sin\theta)^2 d\phi^2\right] \tag{59}$$

at ρ > rS with the definition of the new radial variable of the components of the metric tensor as

$$N(\rho) = \frac{1 - \frac{r_S}{4\rho}}{1 + \frac{r_S}{4\rho}}, \tag{60a}$$

$$\Psi(\rho) = 1 + \frac{r_S}{4\rho}. \tag{60b}$$

The Schwarzschild coordinates and the isotropic coordinates are transformed as

$$r = \rho\left(\rho - + \frac{r_S}{4\rho}\right)^2 \tag{61}$$

or $$2\rho = r - \frac{r_S}{2} + \sqrt{r^2 - r r_S} \tag{62}$$

From [10][48], one has that the expression of the laspe function is retrieved as

$$N_\epsilon\sqrt{\left(\rho - \frac{r_S}{4}\right)^2 + \epsilon^2}. \tag{63}$$

The Schwarzschild fictitious singularity is located at

$$\rho = \frac{r_S}{4}. \tag{64}$$

The photon sphere is situated at ρ_{ps} as

$$\rho_{ps} = (2 + \sqrt{3})\frac{r_S}{4}. \tag{65}$$

Theradial part of the line element Eq. (59) is regular at the fictitious singularity; there appears physical singularities in the presence of matter fields as

$$lim_{r\to r_S}\sqrt{-g} = N\Psi^6\rho^2 sin\theta = 0. \tag{66}$$

The ingredients are ready for the analysis of the 't Hooft brick-wall model [51]: the exploration of the pscaetime at this layer, i.e.

$$rs < r < r_S + \epsilon \qquad (67)$$

in the strong field regime can be performed.

8. Discussion

More in detail, from [51], blackhole objects are commented to be treated as 'composite sysyems' of elementary particles.’ This analysis is to be reconducted to the notion of 'general coordinate transformation invariance: the task was stressed not to be achieved yet.

Results from foliated manifolds can be also used for the present purposes. From [35], the procedure is shown, for which it is possible to construct a non-trivial killing tensor of rank 2 on a foliated manifold of codimension 1. The non-trivial Killing tensors can also be 'lifted' from the slices to the entire manifold. As a result, from [52], the question can be examined, about whether there exist generalized photon surfaces in a spacetime with two killing vector fields.

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