Covariance Projection Filter

Subjects: Engineering, Electrical & Electronic

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Consider n data sources with their data represented, respectively, by N_1, N_2,...,N_n dimensional vectors, such that $\sum_{i=1}^{n} N_i = N_0$. The covariance projection framework of data fusion, or, shortly, the covariance projection filter (CPF), is based on projecting the joint probability distribution of those N_0 variables from n data sources onto the constraint manifold formed in the N_0 dimensional space, referred to here as the extended space, based on the constraints present among the N variables. Then, the covariance projection framework of data fusion represents the projected probability distribution on the constraint manifold as the result of data fusion. For instance, in CPF, the fused data can be chosen as the point on the constraint manifold that bears the maximum probability, while the uncertainty associated with the resulting fused data can be defined as the probability distribution around the chosen point on the constraint manifold. The covariance projection framework of data fusion was initially conceived by Sukhan Lee and further elaborated into a more formal mathematical discipline by Sukhan Lee and Muhammad Abu Bakr. It turns out that CPF is equivalent to other well-known data fusion methods such as Kalman filter, Bar Shalom Campo and generalized Millman's formula for linear systems with known Gaussian noise either uncorrelated or correlated. However, CPF provides a general framework of data fusion that allows incorporation of any system constraints as well as detection of data inconsistency directly into data fusion, besides opening a new possibility of handling non-linear systems with non-Gaussian noise. In what follows, the mathematical formula of CPF developed, in particular, for a linear Gaussian system with linear constraint is introduced.

Bar-Shalom Campo | Covariance Projection method | data fusion | distributed architecture | Kalman filter | linear constraints | inconsistent data

1. Introduction

The method [1][2] first represents the probability of true states and measurements in the extended space around the data from state predictions and sensor measurements, where the extended space is formed by taking states and measurements as independent variables. Any constraints among true states and measurements that should be satisfied are then represented as a constraint manifold in the extended space. This is shown schematically in Figure 1a for filtering as an example. Data fusion is accomplished by projecting the probability distribution of true states and measurements onto the constraint manifold.

More specifically, consider two mean estimates, $\hat{x}_1$ and $\hat{x}_2$, of the state $x \in R^N$, with their respective covariances as P_1, $P_2 \in R^{N\times N}$. Furthermore, the estimates are assumed to be correlated with cross-covariance P_{12}. The mean estimates and their covariances together with their cross-covariance in R^N are then transformed to an extended space of R^{2N} along with the linear constraint between the two estimates:

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & P_{12} \\ P_{21}^T & P_2 \end{bmatrix}, \quad C_1\hat{x}_1 = C_2\hat{x}_2 \tag{1}$$

where C_1 and C_2 are constant matrices of compatible dimensions. In the case where $\hat{x}_1$ and $\hat{x}_2$ estimate the same entity, C_1 and C_2 become identity matrix I. Figure 1b illustrates schematically the fusion of $\hat{x}_1$ and $\hat{x}_2$ in the extended space based on the proposed method. Fusion takes place by finding the point on the constraint manifold that represents the minimum weighted distance from $\hat{x}$ in R^{2N}, where the weight is given by P.

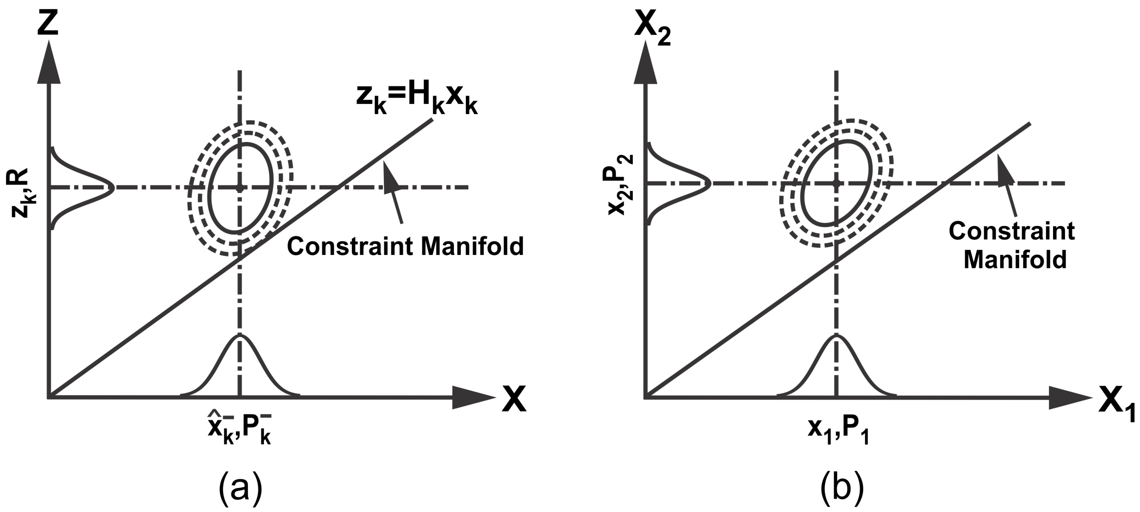

Figure 1. (a) Probability of true states and measurements in the extended space around the data from state predictions and sensor measurements and constraint manifold (b) Extended space representation of two data sources with constraint manifold.

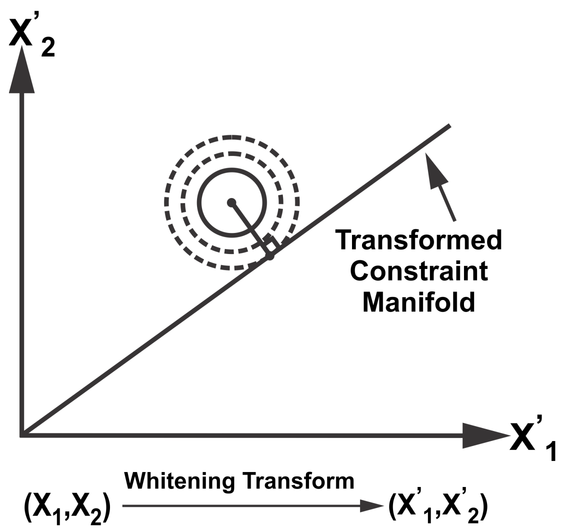

Figure 2. Whitening transform and projection.

To find a point on the constraint manifold with minimum weighted distance, we apply the whitening transform (WT) defined as, $W = D^{-1/2}E^T$, where D and E are the eigenvalue and eigenvector matrices of P. Applying WT,

$$\hat{x}^W = W\hat{x}, \quad P^W = WPW^T, \quad M^W = WM$$

where the matrix $M = [C_1 \quad C_2]^T$ is the subspace of the constraint manifold. Figure 2 illustrates the transformation of the probability distribution as an ellipsoid into a unit circle after WT. The probability distribution is then orthogonally projected on the transformed manifold M^W to satisfy the constraints between the data sources in the transformed space as illustrated in Figure 2. Inverse WT is applied to obtain the fused mean estimate and covariance in the original space,

$$\tilde{x} = W^{-1}P_r W\hat{x} \qquad (2)$$

$$\widetilde{P} = W^{-1}P_r P_r^T W^{-T} \qquad (3)$$

where $P_r = M^W(M^{W^T}M^W)^{-1}M^{W^T}$ is the orthogonal projection matrix. Using the definition of various components in (2) and (3), a close form simplification can be obtained as,

$$\tilde{x} = M(M^T P^{-1} M)^{-1} M^T P^{-1} \hat{x} \qquad (4)$$

$$\widetilde{P} = M(M^T P^{-1} M)^{-1} M^T \qquad (5)$$

Due to the projection in extended space of R^{2N}, (4) and (5) provide a fused result with respect to each data source. In the case where $\hat{x}_1$ and $\hat{x}_2$ estimate the same entity, that is, $M = [I_N \quad I_N]^T$, the fused result will be same for the two data sources. As such, a close form equation for fusing redundant data sources in R^N can be obtained from (4) and (5) as,

$$\tilde{x} = (M^T P^{-1} M)^{-1} M^T P^{-1} \hat{x} \qquad (6)$$

$$\widetilde{P} = (M^T P^{-1} M)^{-1} \qquad (7)$$

Given n mean estimates $\hat{x}_1, \hat{x}_2, .., \hat{x}_n$ of a state $x \in R^N$ with their respective covariances $P_1, P_2, \ldots, P_n \in R^{N\times N}$ and known cross-covariances $P_{ij}, i, j = 1, \ldots, n$, (6) and (7) can be used to obtain the optimal fused mean estimate and covariance with $M = [I_{N1} \quad I_{N2} \quad \ldots \quad I_{Nn}]^T$.

For fusing correlated estimates from n redundant sources, the CPF is equivalent to the weighted fusion algorithms [3][4], which compute the fused mean estimate and covariance as a summation of weighted individual estimates as,

$$\tilde{x} = \sum_{i=1}^{n} c_i \hat{x}_i \,, \qquad \widetilde{P} = \sum_{i,j=1}^{n} c_i P_{ij} c_j^T \tag{8}$$

with $\sum_{i=1}^{n} c_i = I$. Equivalently, the CP fused mean and covariance can be written as,

$$\tilde{x} = L\hat{x} \,, \qquad \widetilde{P} = LPL^T \tag{9}$$

where $L = [\,L_1 \quad L_2 \quad \ldots \quad L_n\,] = (M^T P^{-1} M)^{-1} M^T P^{-1}$ and $\sum_{i=1}^{n} L_i = I$. In the particular case of two data sources, the CP fused solution reduces to the well-known Bar-Shalom Campo formula [5],

$$\tilde{x} = (P_2 - P_{21})(P_1 + P_2 - P_{12} - P_{21})^{-1}\hat{x}_1 + (P_1 - P_{12})(P_1 + P_2 - P_{12} - P_{21})^{-1}\hat{x}_2 \tag{10}$$

$$\widetilde{P} = P_1 - (P_1 - P_{12})(P_1 + P_2 - P_{12} - P_{21})^{-1}(P_1 - P_{21}) \tag{11}$$

Although equivalent to the traditional approaches in fusing redundant data sources, the proposed method offers a generalized framework not only for fusing correlated data sources but also for handling linear constraints and data inconsistency simultaneously within the framework.

2. Detection of Data Inconsistency

The proposed approach exploits the constraint manifold among sensor estimates to identify any data inconsistency. The identification of inconsistent data is based on the distance from the constraint manifold to the mean of redundant data sources in the extended space that provides a confidence measure with the relative disparity among data sources. Assuming a joint multivariate normal distribution for the data sources, the data confidence can be measured by computing the distance from the constraint manifold as illustrated in Figure 3.

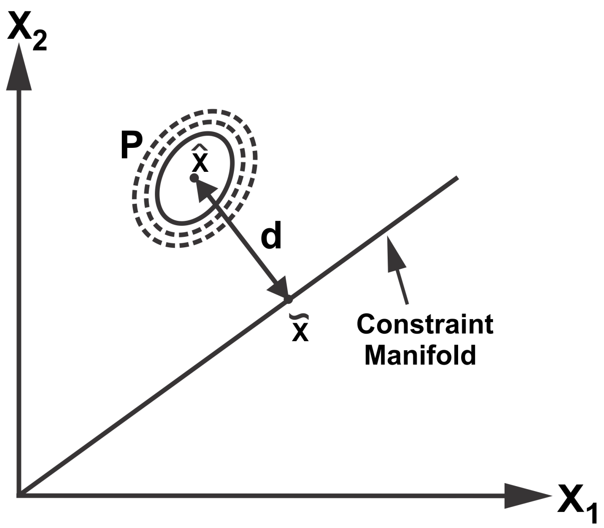

Figure 3. The distance of the multi-variate distribution from the constraint manifold.

Consider the joint space representation of n sensor estimates ,

$$\hat{x} = \begin{bmatrix} x_{N1} \\ x_{N2} \\ \vdots \\ x_{Nn} \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & P_{12} & \cdots & P_{1n} \\ P_{12}^T & P_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ P_{1n}^T & \cdots & \cdots & P_{nn} \end{bmatrix}$$

where N is the dimension of the state vector. The distance d can be computed as,

$$d = (\hat{x} - \tilde{x})^T P^{-1} (\hat{x} - \tilde{x}) \quad (12)$$

where $\tilde{x}$ is the point on the manifold and can be obtain by using (4). The d distance follows a chi-square distribution with N_0 degrees of freedom (DOF), that is, $d \sim \chi^2(N_0)$. A chi-square table is then used to obtain the critical value for a particular significance level and DOF. A computed distance d less than the critical value mean that we are confident about the closeness of sensor estimates and that they can be fused together to provide a better estimate of the underlying states. On the other hand, a distance d greater than or equal to the critical value indicate spuriousness of the sensor estimates.

3. Incorporation of Linear Constraints

Consider a linear dynamic system model,

$$x_k = A_{k-1} x_{k-1} + B_{k-1} u_{k-1} + w_{k-1} \quad (13)$$

$$z_{k_i} = H_{k_i} x_k + v_{k_i}, i = 1, \ldots, n \quad (14)$$

where k represents the discrete-time index, A_k is the system matrix, B_k is the input matrix, u_k is the input vector and x_k is the state vector. The system process noise w_k with covariance matrix Q and measurement noise v_k with covariance R are assumed to be correlated with cross-covariance P_{QR}. The state $x_k \in R^N$ is known to be constrained as,

$$Cx_k = c = 0 \quad (15)$$

For $c \neq 0$, the state space can be translated by a factor c such that $C\bar{x} = 0$. After constrained state estimation, the state space can be translated back by the factor c to satisfy $Cx_k = c$. Hence, without loss of generality, the $c = 0$ case is considered for analysis here. The matrix $C \in R^{n \times m}$ is assumed to have a full row rank.

The CPF incorporates any linear constraints among states without any additional processing. Let us denote the constrained filtered estimate of the CPF as $(\hat{x}^c, P^c)$. Assume $(\hat{x}_k{}^-, P_k{}^-)$ as the predicted state estimate based on the underlying system equation. The extended space representation of the state predictions and measurements of multiple sensors can be written as,

$$\hat{x} = \begin{bmatrix} \hat{x}_k^{-} \\ z_{k_1} \\ \vdots \\ z_{k_n} \end{bmatrix}, \quad P = \begin{bmatrix} P_k^{-} & P_{P_k^- R_1} & \cdots & P_{P_k^- R_n} \\ P_{P_k^- R_1}^T & R_1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ P_{P_k^- R_n}^T & \cdots & \cdots & R_n \end{bmatrix}$$

Then the CPF estimate in the presence of linear constraints among states can be obtained using (4) and (5) as,

$$\hat{x}^c = M_c (M_c^T P^{-1} M_c)^{-1} M_c^T P^{-1} \hat{x} \quad (16)$$

$$P^c = M_c (M_c^T P^{-1} M_c)^{-1} M_c^T \quad (17)$$

where the M_c matrix is the subspace of the constraint among the state prediction x_k^- and sensor measurements z_{k_i} as well as linear constraints C among state variables. The subspace of the linear constraint among state prediction and sensor measurements can be written as,

$$M = [\, I_N, \quad H_1, \quad \cdots, \quad H_n \,]^T$$

Then, M_c is a combination of M and C, that is,

$$M_c \in (M, C)$$

The projection of the probability distribution of true states and measurements around the predicted states and actual measurements onto the constraint manifold M_c in the extended space provide the filtered or fused estimate of state prediction and sensor measurements as well as completely satisfying the linear constraints among the states directly in one step.

References

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