Oscillatory Properties of Noncanonical Neutral DDEs of Second-Order

Subjects: Mathematics

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A DDE is a single-variable differential equation, usually called time, in which the derivative of the solution at a certain time is given in terms of the values of the solution at earlier times. Moreover, if the highest-order derivative of the solution appears both with and without delay, then the DDE is called of the neutral type. The neutral DDEs have many interesting applications in various branches of applied science, as these equations appear in the modeling of many technological phenomena. The problem of studying the oscillatory and nonoscillatory properties of DDEs has been a very active area of research in the past few decades.

delay differential equation neutral oscillation noncanonical case

1. Introduction

Consider the 2nd-order delay differential equation (DDE) of the neutral type:

$$\left(a_{0}\left(t
ight)\left(v'\left(t
ight)
ight)^{eta}
ight)'+a_{2}\left(t
ight)u^{eta}\left(g_{1}\left(t
ight)
ight)=0,$$

(1)

where $t \in [t_0, \infty)$ and $v(t) := u(t) + a_1(t)u(g_0(t))$. In this entry, we obtain new sufficient criteria for the oscillation of solutions of (1) under the following hypotheses:

(A1) $\beta \geq 1$ is a ratio of odd integers;

(A2) $a_i \in C([t_0, \infty), [0, \infty))$ for $i = 0, 1, 2, a_0(t) > 0$ $a_1 \le c_0$ a constant (this constant plays an important role in the results), and a_2 does not vanish identically on any half-line $[t_*, \infty)$ with $t_* \in [t_0, \infty)$;

(A3)
$$g_j\in C([t_0,\infty),\mathbb{R})$$
 $g_j(t)\leq t$ $g_0'(t)\geq g_0^*>0,$ $g_0\circ g_1=g_1\circ g_0$ and $\lim_{t o\infty}g_j(t)=\infty$ for $j=0,1$

By a proper solution of (1), we mean a $u \in C^1([t_0,\infty))$ with $a_0 \cdot (v')^{\beta} \in C^1([t_0,\infty))$ and $\sup \{|u(t)| : t \ge t_*\} > 0$, for $t_* \in [t_0,\infty)$, and u satisfies (1) on $[t_0,\infty)$. A solution u of (1) is called nonoscillatory if it is eventually positive or eventually negative; otherwise, it is called oscillatory.

The oscillatory properties of solutions of second-order neutral DDE $(\underline{1})$ in the noncanonical case, that is:

$$\eta\left(t_{0}
ight)<\infty,$$

(2)

where

$$\eta\left(t
ight):=\int_{t}^{\infty}a_{0}^{-1/eta}\left(\mu
ight)\mathrm{d}\mu.$$

2. Oscillatory Properties of Noncanonical Neutral DDEs of Second-Order

We begin with the following notations: U^+ is the set of all eventually positive solutions of (1), $V(t):=a_0^{1/\beta}(t)v'(t)$,

Lemma 1. Assume that $v\in U^+$ and there exists a $\,\delta_0\in(0,1)$ such that:

$$ilde{a}_2\left(t
ight)a_0^{1/eta}\left(t
ight)\eta^{eta+1}\left(t
ight)\geq \delta_0.$$

Then, v eventually satisfies:

(C₁) v is decreasing and converges to zero;
(C₂)
$$v(t) \ge -\eta(t)V(t)$$
 and $\frac{v}{\eta}$ is increasing,

and:

$$egin{aligned} (\mathbf{C}_3) & V'\left(t
ight) + rac{c_0^eta}{g_0^*} (V\left(g_0\left(t
ight)
ight))' + rac{2^{1-eta}}{eta} \eta^{eta-1}\left(t
ight) ilde{a}_2\left(t
ight) v\left(g_1\left(t
ight)
ight) \leq 0. \end{aligned}$$

Proof. Let $u \in U^+$. Then, we have that $u(g_0(t))$, and $u(g_1(t))$ are positive for $t \ge t_1$, for some $t_1 \ge t_0$. Therefore, it follows from (<u>1</u>) that:

Using $(\underline{1})$ and Lemma 1 in $[\underline{1}]$, we see that:

$$\begin{array}{lcl} 0 & = & \left(V^{\beta}\left(t\right)\right)' + \frac{c_{0}^{\beta}}{g_{0}'\left(t\right)} \left(V^{\beta}\left(g_{0}\left(t\right)\right)\right)' + a_{2}\left(t\right) u^{\beta}\left(g_{1}\left(t\right)\right) \\ & & + c_{0}^{\beta} a_{2}\left(g_{0}\left(t\right)\right) u^{\beta}\left(g_{1}\left(g_{0}\left(t\right)\right)\right) \\ & \geq & \left(V^{\beta}\left(t\right)\right)' + \frac{c_{0}^{\beta}}{g_{0}^{*}} \left(V^{\beta}\left(g_{0}\left(t\right)\right)\right)' + \tilde{a}_{2}\left(t\right) \left[u^{\beta}\left(g_{1}\left(t\right)\right) + c_{0}^{\beta} u^{\beta}\left(g_{0}\left(g_{1}\left(t\right)\right)\right)\right] \\ & \geq & \left(V^{\beta}\left(t\right) + \frac{c_{0}^{\beta}}{g_{0}^{*}} V^{\beta}\left(g_{0}\left(t\right)\right)\right)' + 2^{1-\beta} \tilde{a}_{2}\left(t\right) [u\left(g_{1}\left(t\right)\right) + c_{0} u\left(g_{0}\left(g_{1}\left(t\right)\right)\right)]^{\beta} \end{array}$$

and so:

$$\left(V^{eta}\left(t
ight)+rac{c_{0}^{eta}}{g_{0}^{st}}V^{eta}\left(g_{0}\left(t
ight)
ight)
ight)^{\prime}+2^{1-eta} ilde{a}_{2}\left(t
ight)v^{eta}\left(g_{1}\left(t
ight)
ight)\leq0.$$

(4)

Integrating this inequality from t_1 to t and using the fact $\, \left(V^{eta}(t)
ight)' \leq 0$, we find:

$$ilde{c}_0 V^eta\left(t
ight) \leq ilde{c}_0 V^eta\left(g_0\left(t_1
ight)
ight) - 2^{1-eta}\int_{t_1}^t ilde{a}_2\left(\mu
ight) v^eta\left(g_1\left(\mu
ight)
ight) \mathrm{d}\mu.$$

(5)

 $({f C}_1)$ Assume the contrary, that v'(t)>0 for $t\geq t_1$. Thus, from (5), we have:

$$V^{\beta}\left(t\right) \leq V^{\beta}\left(g_{0}\left(t_{1}\right)\right) - \frac{2^{1-\beta}}{\tilde{c}_{0}}v^{\beta}\left(g_{1}\left(t_{1}\right)\right)\int_{t_{1}}^{t}\tilde{a}_{2}\left(\mu\right)\mathrm{d}\mu.$$

This, from (<u>3</u>), implies:

$$egin{aligned} V^eta\left(t
ight) &\leq V^eta\left(g_0\left(t_1
ight)
ight) - rac{2^{1-eta}}{ ilde{c}_0}\delta_0v^eta\left(g_1\left(t_1
ight)
ight) \int_{t_1}^t rac{1}{a_0^{1/eta}\left(\mu
ight)\eta^{eta+1}\left(\mu
ight)}\mathrm{d}\mu \ &\leq V^eta\left(g_0\left(t_1
ight)
ight) - \gamma_0\delta_0v^eta\left(g_1\left(t_1
ight)
ight) \left(rac{1}{\eta^eta\left(t
ight)} - rac{1}{\eta^eta\left(t_1
ight)}
ight). \end{aligned}$$

Letting $t \to \infty$ and taking the fact that $\eta(t) \to 0$ as $t \to \infty$, we obtain $V^{\beta}(t) \to -\infty$, which contradicts the positivity of V(t).

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Next, since v is positive decreasing, we have that $\lim_{t\to\infty} v(t) = v_0 \ge 0$. Assume the contrary, that $v_0 > 0$. Then, $v(t) \ge v_0$ for all $t \ge t_2$, for some $t_2 \ge t_1$. Thus, from (3) and (5), we have:

$$egin{aligned} V^eta\left(t
ight) &\leq V^eta\left(g_0\left(t_1
ight)
ight) - rac{2^{1-eta}}{ ilde{c}_0}v_0^eta\int_{t_1}^t ilde{a}_2\left(\mu
ight)\mathrm{d}\mu\ &\leq -2^{1-eta}eta\delta_0v_0^eta\int_{t_1}^t rac{1}{a_0^{1/eta}\left(\mu
ight)\eta^{eta+1}\left(\mu
ight)}\mathrm{d}\mu\ &\leq -\gamma_0\delta_0v_0^eta\left(rac{1}{\eta^eta\left(t
ight)} - rac{1}{\eta^eta\left(t_1
ight)}
ight), \end{aligned}$$

or

$$v'\left(t
ight)\leq-\gamma_{0}^{1/eta}\delta_{0}^{1/eta}v_{0}rac{1}{a_{0}^{1/eta}\left(t
ight)}igg(rac{1}{\eta^{eta}\left(t
ight)}-rac{1}{\eta^{eta}\left(t_{1}
ight)}igg)^{1/eta},$$

and so,

$$v'\left(t
ight)\leq-\gamma_{0}^{1/eta}\delta_{0}^{1/eta}v_{0}rac{1}{a_{0}^{1/eta}\left(t
ight)\eta\left(t
ight)}igg(1-rac{\eta^{eta}\left(t
ight)}{\eta^{eta}\left(t_{1}
ight)}igg)^{1/eta}.$$

(6)

Using the fact that $\eta'(t) < 0$, we obtain that $\eta(t) < \eta'(t_2) < \eta'(t_1)$ for all $t \ge t_2 \ge t_1$. Hence, by integrating (6) from t_1 to t, we obtain:

$$egin{aligned} & v\left(t
ight) &\leq v\left(t_{2}
ight) - \gamma_{0}^{1/eta}\delta_{0}^{1/eta}v_{0}\int_{t_{2}}^{t}rac{1}{a_{0}^{1/eta}\left(\mu
ight)\eta\left(\mu
ight)}\left(1-rac{\eta^{eta}\left(\mu
ight)}{\eta^{eta}\left(t_{1}
ight)}
ight)^{1/eta}\mathrm{d}\mu \ &\leq v\left(t_{2}
ight) - \gamma_{0}^{1/eta}\delta_{0}^{1/eta}v_{0}\left(1-rac{\eta^{eta}\left(t_{2}
ight)}{\eta^{eta}\left(t_{1}
ight)}
ight)^{1/eta}\int_{t_{2}}^{t}rac{1}{a_{0}^{1/eta}\left(\mu
ight)\eta\left(\mu
ight)}\mathrm{d}\mu \ &\leq v\left(t_{2}
ight) - \gamma_{0}^{1/eta}\delta_{0}^{1/eta}v_{0}\left(1-rac{\eta^{eta}\left(t_{2}
ight)}{\eta^{eta}\left(t_{1}
ight)}
ight)^{1/eta}\lnrac{\eta\left(t_{2}
ight)}{\eta\left(t
ight)}. \end{aligned}$$

Letting $t \to \infty$ and taking the fact that $\eta(t) \to 0$ as $t \to \infty$, we obtain $v(t) \to -\infty$, which contradicts the positivity of v(t). Therefore, $v_0 = 0$.

 (\mathbf{C}_2) Since V(t) is decreasing, we obtain:

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$$egin{aligned} &-a_0^{-1/eta}\left(t
ight)v\left(t
ight) &\leq a_0^{-1/eta}\left(t
ight)\int_t^\infty a_0^{-1/eta}\left(\mu
ight)V\left(\mu
ight)\mathrm{d}\mu\ &\leq a_0^{-1/eta}\left(t
ight)V\left(t
ight)\int_t^\infty a_0^{-1/eta}\left(\mu
ight)\mathrm{d}\mu \end{aligned}$$

and:

$$-a_{0}^{-1/eta}\left(t
ight)v\left(t
ight)\leq v'\left(t
ight)\eta\left(t
ight).$$

(7)

Then, $(v/\eta)' \geq 0$.

 (\mathbf{C}_3) From (<u>7</u>), we obtain:

$$-rac{v\left(g\left(t
ight)
ight)}{\eta\left(t
ight)}\leq-rac{v\left(t
ight)}{\eta\left(t
ight)}\leq V\left(t
ight).$$

Thus, from (4) and the fact $\,V'(t)\leq 0$, we obtain:

$$eta V^{eta-1}\left(t
ight)V'\left(t
ight)+rac{c_{0}^{eta}}{g_{0}^{*}}eta V^{eta-1}\left(g_{0}\left(t
ight)
ight)\left(V\left(g_{0}\left(t
ight)
ight)
ight)'+2^{1-eta} ilde{a}_{2}\left(t
ight)v^{eta}\left(g_{1}\left(t
ight)
ight)\leq0,$$

and then:

$$V'\left(t
ight)+rac{c_{0}^{eta}}{g_{0}^{st}}(V\left(g_{0}\left(t
ight)
ight))'+rac{2^{1-eta}}{eta}\eta^{eta-1}\left(t
ight) ilde{a}_{2}\left(t
ight)v\left(g_{1}\left(t
ight)
ight)\leq0.$$

The proof is complete.

Lemma 2. Assume that $u\in U^+$ and there exists a $\delta_0\in(0,1)$ such that (3) holds. Then:

$$(\mathbf{C}_4) \hspace{0.2cm} \eta \left(t
ight) V \left(t
ight) \leq - \gamma_0 \delta_0 v \left(t
ight) \hspace{0.2cm} and \hspace{0.2cm} v / \eta^{\gamma_0 \delta_0} \hspace{0.2cm} is \hspace{0.2cm} decreasing.$$

Proof. Let $u\in U^+$. From Lemma 1, we have that $({f C}_1)$ – $({f C}_3)$ hold for $t\geq t_1$.

Integrating (\mathbf{C}_3) from t_1 to t, we arrive at:

$$V\left(t
ight)\leq V\left(g_{0}\left(t_{1}
ight)
ight)-\gamma_{0}\int_{t_{1}}^{t}\eta^{eta-1}\left(\mu
ight) ilde{a}_{2}\left(\mu
ight)v\left(g_{1}\left(\mu
ight)
ight)\mathrm{d}\mu.$$

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From $(\underline{3})$, we obtain:

$$V\left(t
ight)\leq V\left(g_{0}\left(t_{1}
ight)
ight)-\gamma_{0}\delta_{0}v\left(t
ight)\int_{t_{1}}^{t}rac{1}{a_{0}^{1/eta}\left(\mu
ight)\eta^{2}\left(\mu
ight)}\mathrm{d}\mu,$$

and:

$$V\left(t
ight)\leq V\left(g_{0}\left(t_{1}
ight)
ight)+\gamma_{0}\delta_{0}v\left(t
ight)\left(rac{1}{\eta\left(t_{1}
ight)}-rac{1}{\eta\left(t
ight)}
ight).$$

(8)

Using (\mathbf{C}_1) , we eventually have:

$$V\left(g_{0}\left(t_{1}
ight)
ight)+\gamma_{0}\delta_{0}rac{v\left(t
ight)}{\eta\left(t_{1}
ight)}\leq0,$$

Hence, (8) becomes:

This implies that $v/\eta^{\gamma_0\delta_0}$ is a decreasing function. The proof is complete.

2.2. Oscillation Theorems

In the next theorem, by using the principle of comparison with an equation of the first-order, we obtain a new criterion for the oscillation of $(\underline{1})$.

Theorem 6. Assume that $g_1(t) \le g_0(t)$ and there exists a $\delta_0 \in (0, 1)$ such that (3) holds. If the delay differential equation:

$$W^{\prime}\left(t
ight)+rac{\gamma_{0}}{\left(1-\gamma_{0}\delta_{0}
ight)}\eta^{eta}\left(t
ight) ilde{a}_{2}\left(t
ight)W\left(g_{0}^{-1}\left(g_{1}\left(t
ight)
ight)
ight)=0$$

(9)

is oscillatory, then every solution of $(\underline{1})$ is oscillatory.

Proof. Assume the contrary, that (1) has a solution $u \in U^+$. Then, we have that u(t), and $u(g_1(t))$ are positive for $t \ge t_1$, for some $t_1 \ge t_0$. From Lemmas 1 and 2, we have that $(\mathbf{C}_1) - (\mathbf{C}_4)$ hold for $t \ge t_1$.

Next, we define:

$$w\left(t
ight):=\eta\left(t
ight)V\left(t
ight)+v\left(t
ight).$$

From $({f C}_1)$, w(t)>0 for $t\geq t_1$. Thus,

$$w'(t) = \eta(t)V'(t) \le 0.$$

Thus, it follows from (\mathbf{C}_3) that:

$$w'(t) + rac{c_0^eta}{g_0^*} (w(g_0(t)))' + rac{2^{1-eta}}{eta} \eta^eta(t) ilde{a}_2(t) v(g_1(t)) \le 0.$$

Using (\mathbf{C}_4) , we obtain that:

$$egin{array}{rcl} w\left(t
ight) &=& \eta\left(t
ight)V\left(t
ight)+v\left(t
ight) \ &\leq& -\gamma_{0}\delta_{0}v\left(t
ight)+v\left(t
ight) \ &=& (1-\gamma_{0}\delta_{0})v\left(t
ight), \end{array}$$

which with (10) gives:

$$w'(t) + \frac{c_0^{\beta}}{g_0^*} (w(g_0(t)))' + \frac{2^{1-\beta}}{\beta(1-\gamma_0\delta_0)} \eta^{\beta}(t)\tilde{a}_2(t)w(g_1(t)) \le 0.$$
(11)

Now, we set:

$$W\left(t
ight):=w\left(t
ight)+rac{c_{0}^{eta}}{g_{0}^{st}}w\left(g_{0}\left(t
ight)
ight)>0.$$

Then, $W(t) \leq ilde{c}_0 w(g_0(t))$, and so, (11) becomes:

which has a positive solution. In view of ^[2] (Theorem 1), (9) also has a positive solution, which is a contradiction. The proof is complete.

Corollary 1. Assume that $g_1(t) \leq g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If:

$$\lim_{t\to\infty}\sup_{t\to\infty}\;\int_{t_1}^t\left(\frac{2^{1-\beta}}{\beta}\eta^\beta\left(\mu\right)\tilde{a}_2\left(\mu\right)\frac{\eta^{\gamma_0\delta_0}\left(g_1\left(\mu\right)\right)}{\eta^{\gamma_0\delta_0}\left(g_0\left(\mu\right)\right)}-\frac{\hat{c}_0}{4}\frac{1}{a_0^{1/\beta}\left(g_0\left(\mu\right)\right)\eta\left(\mu\right)}\right)\mathrm{d}\mu=\infty,$$

(12)

then every solution of $(\underline{1})$ is oscillatory.

Proof. It follows from Theorem 2 in [3] that the condition $(\underline{12})$ implies the oscillation of $(\underline{9})$.

Next, by introducing two Riccati substitution, we obtain a new oscillation criterion for $(\underline{1})$.

Theorem 7. Assume that $g_1(t) \leq g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If:

$$\lim_{t o\infty} \sup_{t o\infty} \ \int_{t_1}^t \left(rac{2^{1-eta}}{eta} \eta^eta\left(\mu
ight) ilde{a}_2\left(\mu
ight) rac{\eta^{\gamma_0\delta_0}\left(g_1\left(\mu
ight)
ight)}{\eta^{\gamma_0\delta_0}\left(g_0\left(\mu
ight)
ight)} - rac{\hat{c}_0}{4} rac{1}{a_0^{1/eta}\left(g_0\left(\mu
ight)
ight) \eta\left(\mu
ight)}
ight) \mathrm{d}\mu = \infty,$$

(13)

then every solution of $(\underline{1})$ is oscillatory.

Proof. Assume the contrary, that (<u>1</u>) has a solution $u \in U^+$. Then, we have that u(t), and $u(g_1(t))$ are positive for $t \ge t_1$, for some $t_1 \ge t_0$. From Lemmas 1 and 2, we have that $(\mathbf{C}_1) - (\mathbf{C}_4)$ hold for $t \ge t_1$.

Now, we define the functions:

$$\Theta_1 := \frac{V}{v},\tag{1}$$

and:

$$\Theta_2 := \frac{V \circ g_0}{v \circ g_0}.$$
(2)

Then, Θ_1 and Θ_2 are negative for $t \geq t_1$. From $({f C}_4)$, we obtain:

$$rac{v\circ g_1}{\eta^{\gamma_0\delta_0}\circ g_1}\geq rac{v\circ g_0}{\eta^{\gamma_0\delta_0}\circ g_0}\geq rac{v}{\eta^{\gamma_0\delta_0}}.$$

Hence,

$$egin{aligned} \Theta_1' &=& rac{V'}{v} - rac{V}{v^2} v' = rac{V'}{v \circ g_1} rac{v \circ g_1}{v} - rac{1}{a^{1/eta}} iggl(rac{V}{v}iggr)^2 \ &\leq& rac{\eta^{\gamma_0 \delta_0} \circ g_1}{\eta^{\gamma_0 \delta_0}} rac{V'}{v \circ g_1} - rac{1}{a^{1/eta}} \Theta_1^2, \ &\leq& rac{\eta^{\gamma_0 \delta_0} \circ g_1}{\eta^{\gamma_0 \delta_0} \circ g_0} rac{V'}{v \circ g_1} - rac{1}{a^{1/eta}} \Theta_1^2, \end{aligned}$$

and:

$$\begin{split} \Theta_2' &= \frac{\left(V \circ g_0\right)'}{v \circ g_0} - \frac{V \circ g_0}{\left(v \circ g_0\right)^2} \left(v' \circ g_0\right) g_0' \\ &= \frac{\left(V \circ g_0\right)'}{v \circ g_1} \frac{v \circ g_1}{v \circ g_0} - \frac{g_0'}{a^{1/\beta} \circ g_0} \left(\frac{V \circ g_0}{v \circ g_0}\right)^2 \\ &\leq \frac{\eta^{\gamma_0 \delta_0} \circ g_1}{\eta^{\gamma_0 \delta_0} \circ g_0} \frac{\left(V \circ g_0\right)'}{v \circ g_1} - \frac{g_0^*}{\left(a^{1/\beta} \circ g_0\right)} \Theta_2^2. \end{split}$$

Then:

$$\eta\left(t
ight)\Theta_{1}^{\prime}\left(t
ight)-\eta\left(t
ight)rac{\eta^{\gamma_{0}\delta_{0}}\left(g_{1}\left(t
ight)
ight)}{\eta^{\gamma_{0}\delta_{0}}\left(g_{0}\left(t
ight)
ight)}rac{V^{\prime}\left(t
ight)}{v\left(g_{1}\left(t
ight)
ight)}+rac{\eta\left(t
ight)}{a^{1/eta}\left(t
ight)}\Theta_{1}^{2}\left(t
ight)\leq0,$$

(14)

and:

$$\begin{array}{ll} 0 & \geq & \eta \left(g_{0} \left(t \right) \right) \Theta_{2}^{\prime} \left(t \right) - \eta \left(g_{0} \left(t \right) \right) \frac{\eta^{\gamma_{0}\delta_{0}} \left(g_{1} \left(t \right) \right)}{\eta^{\gamma_{0}\delta_{0}} \left(g_{0} \left(t \right) \right)} \frac{\left(V \left(g_{0} \left(t \right) \right) \right)^{\prime}}{v \left(g_{1} \left(t \right) \right)} + \frac{g_{0}^{*} \eta \left(g_{0} \left(t \right) \right)}{a^{1/\beta} \left(g_{0} \left(t \right) \right)} \Theta_{2}^{2} \left(t \right) \\ & \geq & \eta \left(g_{0} \left(t \right) \right) \Theta_{2}^{\prime} \left(t \right) - \eta \left(t \right) \frac{\eta^{\gamma_{0}\delta_{0}} \left(g_{1} \left(t \right) \right)}{\eta^{\gamma_{0}\delta_{0}} \left(g_{0} \left(t \right) \right)} \frac{\left(V \left(g_{0} \left(t \right) \right) \right)^{\prime}}{v \left(g_{1} \left(t \right) \right)} + \frac{g_{0}^{*} \eta \left(g_{0} \left(t \right) \right)}{a^{1/\beta} \left(g_{0} \left(t \right) \right)} \Theta_{2}^{2} \left(t \right). \end{array}$$

(15)

Combining $(\underline{14})$ and $(\underline{15})$, we obtain:

$$\begin{array}{ll} 0 & \geq & \eta \left(g_{0} \left(t \right) \right) \Theta_{2}^{\prime} \left(t \right) - \eta \left(g_{0} \left(t \right) \right) \frac{\eta^{\gamma_{0}\delta_{0}} \left(g_{1} \left(t \right) \right)}{\eta^{\gamma_{0}\delta_{0}} \left(g_{0} \left(t \right) \right)} \frac{\left(V \left(g_{0} \left(t \right) \right) \right)^{\prime}}{v \left(g_{1} \left(t \right) \right)} + \frac{g_{0}^{*} \eta \left(g_{0} \left(t \right) \right)}{a^{1/\beta} \left(g_{0} \left(t \right) \right)} \Theta_{2}^{2} \left(t \right) \\ & \geq & \eta \left(g_{0} \left(t \right) \right) \Theta_{2}^{\prime} \left(t \right) - \eta \left(t \right) \frac{\eta^{\gamma_{0}\delta_{0}} \left(g_{1} \left(t \right) \right)}{\eta^{\gamma_{0}\delta_{0}} \left(g_{0} \left(t \right) \right)} \frac{\left(V \left(g_{0} \left(t \right) \right) \right)^{\prime}}{v \left(g_{1} \left(t \right) \right)} + \frac{g_{0}^{*} \eta \left(g_{0} \left(t \right) \right)}{a^{1/\beta} \left(g_{0} \left(t \right) \right)} \Theta_{2}^{2} \left(t \right). \end{array}$$

Integrating this inequality from t_1 to t, we have:

$$\begin{array}{lll} 0 &\geq & \eta\left(t\right)\Theta_{1}\left(t\right) - \eta\left(t_{1}\right)\Theta_{1}\left(t_{1}\right) + \int_{t_{1}}^{t}\left(a_{0}^{-1/\beta}\left(\mu\right)\Theta_{1}\left(\mu\right) + \frac{\eta\left(\mu\right)}{a^{1/\beta}\left(\mu\right)}\Theta_{1}^{2}\left(t\right)\right) \mathrm{d}\mu \\ &\quad + \frac{c_{0}^{\beta}}{g_{0}^{*}}\left(\eta\left(g_{0}\left(t\right)\right)\Theta_{2}\left(t\right) - \eta\left(g_{0}\left(t_{1}\right)\right)\Theta_{2}\left(t_{1}\right)\right) \\ &\quad + \frac{c_{0}^{\beta}}{g_{0}^{*}}\left(\int_{t_{1}}^{t}a_{0}^{-1/\beta}\left(g_{0}\left(t\right)\right)\Theta_{2}\left(\mu\right) + \frac{g_{0}^{*}\eta\left(g_{0}\left(\mu\right)\right)}{a^{1/\beta}\left(g_{0}\left(\mu\right)\right)}\Theta_{2}^{2}\left(\mu\right)\right) \mathrm{d}\mu \\ &\quad + \frac{2^{1-\beta}}{\beta}\int_{t_{1}}^{t}\eta^{\beta}\left(\mu\right)\tilde{a}_{2}\left(\mu\right)\frac{\eta^{\gamma_{0}\delta_{0}}\left(g_{1}\left(\mu\right)\right)}{\eta^{\gamma_{0}\delta_{0}}\left(g_{0}\left(\mu\right)\right)}\mathrm{d}\mu. \end{array}$$

From $(\mathbf{C}_2),$ we obtain $\eta(t)\Theta_1(t)\geq -1$. Therefore,

where:

$$K:=\eta\left(t_{1}
ight)\Theta_{1}\left(t_{1}
ight)+rac{c_{0}^{eta}}{g_{0}^{st}}\eta\left(g_{0}\left(t_{1}
ight)
ight)\Theta_{2}\left(t_{1}
ight)+\left(1+rac{c_{0}^{eta}}{g_{0}^{st}}
ight).$$

Since $\eta'(t) < 0$ and $a'(t) \geq 0$, we find:

Taking $\mathrm{limsup}_{t \to \infty}$ and using (13), we arrive at a contradiction.

The proof is complete.

2.3. Applications and Discussion

Remark 1. It is easy to see that the previous works that dealt with the noncanonical case required either $a_1(t) < 1$ or $a_1(t) < \eta(t)/\eta(g_0(t))$. Since η is decreasing and $g_0(t) \le t$, we have that $\eta(g_0(t)) \ge \eta(t)$. Then, the results of these works only apply when $a_1(t) \in (0, 1)$.

Example 1. Consider the DDE:

$$\left(t^{2}\left(u\left(t
ight)+a_{1}^{*}u\left(\lambda t
ight)
ight)
ight)^{\prime}+a_{2}^{*}u\left(\kappa t
ight)=0,$$

(16)

where $t\geq 1, a_1^*>0,$ and $\kappa<\lambda\in(0,1)$. By choosing $\delta_0=a_2^*,$ the condition (12) becomes:

$$a_2^*\,\ln\,\frac{\lambda}{\kappa} > \frac{\lambda + a_1^* - \lambda a_2^*}{\mathrm{e}\lambda}$$

(17)

Using Corollary 1, Equation (<u>16</u>) is oscillatory if (<u>17</u>) holds.

Remark 2. To apply Theorems 3 and 4 on (<u>16</u>), we must stipulate that $a_1^* < 1$. Let a special case of (<u>16</u>), namely,

$$\left(t^2\left(u\left(t
ight)+a_1^*u\left(rac{t}{2}
ight)
ight)
ight)'+a_2^*u\left(\kappa t
ight)=0,$$

A simple computation shows that $(\underline{16})$ is oscillatory if:

$$a_{2}^{*}(1-2a_{1}^{*}) > \frac{1}{4} (using Theorem 3)$$
(18)

or:

$$a_{2}^{*}\left(1-2a_{1}^{*}
ight)>1~~(using~Theorem~4)$$

(19)

or:

$$a_2^*\left(1-2a_1^*
ight)\,\ln\,rac{1}{\kappa}>rac{1}{\mathrm{e}}\,\left(using\ Theorem\ 5
ight).$$

Consider the following most specific special case:

$$\left(t^{2}\left(u\left(t\right)+\frac{2}{5}u\left(\frac{t}{2}\right)\right)\right)'+\frac{4}{5}u\left(\frac{t}{4}\right)=0.$$

(21)

Note that $(\underline{18})$ – $(\underline{20})$ fail to apply. However, $(\underline{17})$ reduces to:

$$\frac{4}{5}\ln 2 > \frac{1}{e}$$

which ensures the oscillation of $(\underline{21})$.

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