Oscillatory Properties of Noncanonical Neutral DDEs of Second-Order

Subjects: Mathematics

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A DDE is a single-variable differential equation, usually called time, in which the derivative of the solution at a certain time is given in terms of the values of the solution at earlier times. Moreover, if the highest-order derivative of the solution appears both with and without delay, then the DDE is called of the neutral type. The neutral DDEs have many interesting applications in various branches of applied science, as these equations appear in the modeling of many technological phenomena. The problem of studying the oscillatory and nonoscillatory properties of DDEs has been a very active area of research in the past few decades.

Keywords: delay differential equation ; neutral ; oscillation ; noncanonical case

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1. Introduction

Consider the 2nd-order delay differential equation (DDE) of the neutral type:

$$\left(a_0\,(t)(v'\,(t))^{\beta}\right)' + a_2\,(t)u^{\beta}\,(g_1\,(t)) = 0,$$

(1)

where $t \in [t_0, \infty)$ and $v(t) := u(t) + a_1(t)u(g_0(t))$. In this entry, we obtain new sufficient criteria for the oscillation of solutions of (1) under the following hypotheses:

(A1) $\beta \geq 1$ is a ratio of odd integers;

(A2) $a_i \in C([t_0, \infty), [0, \infty))$ for $i = 0, 1, 2,$ $a_0(t) > 0$ $a_1 \leq c_0$ a constant (this constant plays an important role in the results), and a_2 does not vanish identically on any half-line $[t_*, \infty)$ with $t_* \in [t_0, \infty)$;

(A3) $g_j \in C([t_0, \infty), \mathbb{R})$ $g_j(t) \leq t$ $g_0'(t) \geq \overset{*}{g_0} > 0,$ $g_0 \circ g_1 = g_1 \circ g_0$ and $\lim_{t \to \infty} g_j(t) = \infty$ for $j = 0, 1$

By a proper solution of (1), we mean a $u \in C^1([t_0, \infty))$ with $a_0 \cdot (v')^{\beta} \in C^1([t_0, \infty))$ and $\sup\,\{|u(t)| : t \geq t_*\} > 0,$ for $t_* \in [t_0, \infty)$, and u satisfies (1) on $[t_0, \infty)$. A solution u of (1) is called nonoscillatory if it is eventually positive or eventually negative; otherwise, it is called oscillatory.

The oscillatory properties of solutions of second-order neutral DDE (1) in the noncanonical case, that is:

$$\eta\,(t_0) < \infty,$$

(2)

where

$$\eta\,(t) := \int_t^{\infty} a_0^{-1/\beta}\,(\mu)\mathrm{d}\mu.$$

2. Oscillatory Properties of Noncanonical Neutral DDEs of Second-Order

We begin with the following notations: U^+ is the set of all eventually positive solutions of (1), $V(t) := a_0^{1/\beta}(t)v'(t)$,

Lemma 1. Assume that $v \in U^+$ and there exists a $\delta_0 \in (0, 1)$ such that:

$$\tilde{a}_2(t)a_0^{1/\beta}(t)\eta^{\beta+1}(t) \geq \delta_0.$$

(3)

Then, *v* eventually satisfies:

$$\begin{aligned}&(\mathbf{C}_1) \quad v \textit{ is decreasing and converges to zero};\\&(\mathbf{C}_2) \quad v(t) \geq -\eta(t)V(t) \textit{ and } \frac{v}{\eta} \textit{ is increasing},\end{aligned}$$

and:

$$(\mathbf{C}_3) \quad V'(t) + \frac{c_0^\beta}{g_0^*}(V(g_0(t)))' + \frac{2^{1-\beta}}{\beta}\eta^{\beta-1}(t)\tilde{a}_2(t)v(g_1(t)) \leq 0.$$

Proof. Let $u \in U^+$. Then, we have that $u(g_0(t))$, and $u(g_1(t))$ are positive for $t \geq t_1$, for some $t_1 \geq t_0$. Therefore, it follows from (1) that:

Using (1) and Lemma 1 in [1], we see that:

$$\begin{aligned}0 &= \left(V^\beta(t)\right)' + \frac{c_0^\beta}{g_0'(t)}\left(V^\beta(g_0(t))\right)' + a_2(t)u^\beta(g_1(t))\\&\quad + c_0^\beta a_2(g_0(t))u^\beta(g_1(g_0(t)))\\&\geq \left(V^\beta(t)\right)' + \frac{c_0^\beta}{g_0^*}\left(V^\beta(g_0(t))\right)' + \tilde{a}_2(t)\left[u^\beta(g_1(t)) + c_0^\beta u^\beta(g_0(g_1(t)))\right]\\&\geq \left(V^\beta(t) + \frac{c_0^\beta}{g_0^*}V^\beta(g_0(t))\right)' + 2^{1-\beta}\tilde{a}_2(t)[u(g_1(t)) + c_0u(g_0(g_1(t)))]^\beta\end{aligned}$$

and so:

$$\left(V^\beta(t) + \frac{c_0^\beta}{g_0^*}V^\beta(g_0(t))\right)' + 2^{1-\beta}\tilde{a}_2(t)v^\beta(g_1(t)) \leq 0.$$

(4)

Integrating this inequality from t_1 to t and using the fact $\left(V^\beta(t)\right)' \leq 0$, we find:

$$\tilde{c}_0V^\beta(t) \leq \tilde{c}_0V^\beta(g_0(t_1)) - 2^{1-\beta}\int_{t_1}^t \tilde{a}_2(\mu)v^\beta(g_1(\mu))\mathrm{d}\mu.$$

(5)

$(\mathbf{C}_1)$ Assume the contrary, that $v'(t) > 0$ for $t \geq t_1$. Thus, from (5), we have:

$$V^\beta(t) \leq V^\beta(g_0(t_1)) - \frac{2^{1-\beta}}{\tilde{c}_0}v^\beta(g_1(t_1))\int_{t_1}^t \tilde{a}_2(\mu)\mathrm{d}\mu.$$

This, from (3), implies:

$$\begin{aligned}V^\beta(t) &\leq V^\beta(g_0(t_1)) - \frac{2^{1-\beta}}{\tilde{c}_0}\delta_0v^\beta(g_1(t_1))\int_{t_1}^t \frac{1}{a_0^{1/\beta}(\mu)\eta^{\beta+1}(\mu)}\mathrm{d}\mu\\&\leq V^\beta(g_0(t_1)) - \gamma_0\delta_0v^\beta(g_1(t_1))\left(\frac{1}{\eta^\beta(t)} - \frac{1}{\eta^\beta(t_1)}\right).\end{aligned}$$

Letting $t \to \infty$ and taking the fact that $\eta(t) \to 0$ as $t \to \infty$, we obtain $V^\beta(t) \to -\infty$, which contradicts the positivity of $V(t)$.

Next, since v is positive decreasing, we have that $\lim_{t\to\infty} v(t) = v_0 \geq 0$. Assume the contrary, that $v_0 > 0$. Then, $v(t) \geq v_0$ for all $t \geq t_2$, for some $t_2 \geq t_1$. Thus, from (3) and (5), we have:

$$\begin{aligned} V^{\beta}(t) &\leq V^{\beta}(g_0(t_1)) - \frac{2^{1-\beta}}{\tilde{c}_0} v_0^{\beta} \int_{t_1}^{t} \tilde{a}_2(\mu) \mathrm{d}\mu \\ &\leq -2^{1-\beta}\beta\delta_0 v_0^{\beta} \int_{t_1}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta^{\beta+1}(\mu)} \mathrm{d}\mu \\ &\leq -\gamma_0\delta_0 v_0^{\beta}\left(\frac{1}{\eta^{\beta}(t)} - \frac{1}{\eta^{\beta}(t_1)}\right), \end{aligned}$$

or

$$v'(t) \leq -\gamma_0^{1/\beta}\delta_0^{1/\beta} v_0 \frac{1}{a_0^{1/\beta}(t)} \left(\frac{1}{\eta^{\beta}(t)} - \frac{1}{\eta^{\beta}(t_1)}\right)^{1/\beta},$$

and so,

$$v'(t) \leq -\gamma_0^{1/\beta}\delta_0^{1/\beta} v_0 \frac{1}{a_0^{1/\beta}(t)\eta(t)} \left(1 - \frac{\eta^{\beta}(t)}{\eta^{\beta}(t_1)}\right)^{1/\beta}.$$

(6)

Using the fact that $\eta'(t) < 0$, we obtain that $\eta(t) < \eta'(t_2) < \eta'(t_1)$ for all $t \geq t_2 \geq t_1$. Hence, by integrating (6) from t_1 to *t*, we obtain:

$$\begin{aligned} v(t) &\leq v(t_2) - \gamma_0^{1/\beta}\delta_0^{1/\beta} v_0 \int_{t_2}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta(\mu)} \left(1 - \frac{\eta^{\beta}(\mu)}{\eta^{\beta}(t_1)}\right)^{1/\beta} \mathrm{d}\mu \\ &\leq v(t_2) - \gamma_0^{1/\beta}\delta_0^{1/\beta} v_0 \left(1 - \frac{\eta^{\beta}(t_2)}{\eta^{\beta}(t_1)}\right)^{1/\beta} \int_{t_2}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta(\mu)} \mathrm{d}\mu \\ &\leq v(t_2) - \gamma_0^{1/\beta}\delta_0^{1/\beta} v_0 \left(1 - \frac{\eta^{\beta}(t_2)}{\eta^{\beta}(t_1)}\right)^{1/\beta} \ln \frac{\eta(t_2)}{\eta(t)}. \end{aligned}$$

Letting $t \to \infty$ and taking the fact that $\eta(t) \to 0$ as $t \to \infty$, we obtain $v(t) \to -\infty$, which contradicts the positivity of $v(t)$. Therefore, $v_0 = 0$.

$(\mathbf{C}_2)$ Since $V(t)$ is decreasing, we obtain:

$$\begin{aligned} -a_0^{-1/\beta}(t)v(t) &\leq a_0^{-1/\beta}(t) \int_{t}^{\infty} a_0^{-1/\beta}(\mu)V(\mu) \mathrm{d}\mu \\ &\leq a_0^{-1/\beta}(t)V(t) \int_{t}^{\infty} a_0^{-1/\beta}(\mu) \mathrm{d}\mu \end{aligned}$$

and:

$$-a_0^{-1/\beta}(t)v(t) \leq v'(t)\eta(t).$$

(7)

Then, $(v/\eta)' \geq 0$.

$(\mathbf{C}_3)$ From (7), we obtain:

$$-\frac{v(g(t))}{\eta(t)} \leq -\frac{v(t)}{\eta(t)} \leq V(t).$$

Thus, from (4) and the fact $V'(t) \leq 0$, we obtain:

$$\beta V^{\beta-1}(t)V'(t) + \frac{c_0^{\beta}}{g_0^{*}} \beta V^{\beta-1}(g_0(t))(V(g_0(t)))' + 2^{1-\beta}\tilde{a}_2(t)v^{\beta}(g_1(t)) \leq 0,$$

and then:

$$V'(t) + \frac{c_0^{\beta}}{g_0^{*}}(V(g_0(t)))' + \frac{2^{1-\beta}}{\beta}\eta^{\beta-1}(t)\tilde{a}_2(t)v(g_1(t)) \leq 0.$$

The proof is complete.

Lemma 2. Assume that $u \in U^{+}$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. Then:

$$(\mathbf{C}_4) \quad \eta(t)V(t) \leq -\gamma_0\delta_0 v(t) \quad and \quad v/\eta^{\gamma_0\delta_0} \ is \ decreasing.$$

Proof. Let $u \in U^{+}$. From Lemma 1, we have that $(\mathbf{C}_1)$ – $(\mathbf{C}_3)$ hold for $t \geq t_1$.

Integrating $(\mathbf{C}_3)$ from t_1 to t, we arrive at:

$$V(t) \leq V(g_0(t_1)) - \gamma_0 \int_{t_1}^{t} \eta^{\beta-1}(\mu)\tilde{a}_2(\mu)v(g_1(\mu))\mathrm{d}\mu.$$

From (3), we obtain:

$$V(t) \leq V(g_0(t_1)) - \gamma_0\delta_0 v(t) \int_{t_1}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta^2(\mu)}\mathrm{d}\mu,$$

and:

$$V(t) \leq V(g_0(t_1)) + \gamma_0\delta_0 v(t)\left(\frac{1}{\eta(t_1)} - \frac{1}{\eta(t)}\right).$$

(8)

Using $(\mathbf{C}_1)$, we eventually have:

$$V(g_0(t_1)) + \gamma_0\delta_0 \frac{v(t)}{\eta(t_1)} \leq 0,$$

Hence, (8) becomes:

This implies that $v/\eta^{\gamma_0\delta_0}$ is a decreasing function.
The proof is complete.

2.2. Oscillation Theorems

In the next theorem, by using the principle of comparison with an equation of the first-order, we obtain a new criterion for the oscillation of (1).

Theorem 6. Assume that $g_1(t) \leq g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If the delay differential equation:

$$W'(t) + \frac{\gamma_0}{(1-\gamma_0\delta_0)}\eta^{\beta}(t)\tilde{a}_2(t)W\left(g_0^{-1}(g_1(t))\right) = 0$$

(9)

is oscillatory, then every solution of (1) is oscillatory.

Proof. Assume the contrary, that (1) has a solution $u \in U^{+}$. Then, we have that $u(t)$, and $u(g_1(t))$ are positive for $t \geq t_1$, for some $t_1 \geq t_0$. From Lemmas 1 and 2, we have that $(\mathbf{C}_1)$ – $(\mathbf{C}_4)$ hold for $t \geq t_1$.

Next, we define:

$$w(t) := \eta(t)V(t) + v(t).$$

From $(\mathbf{C}_1)$, $w(t) > 0$ for $t \geq t_1$. Thus,

$$w'(t) = \eta(t)V'(t) \leq 0.$$

Thus, it follows from $(\mathbf{C}_3)$ that:

$$w'(t) + \frac{c_0^{\beta}}{g_0^{*}}(w(g_0(t)))' + \frac{2^{1-\beta}}{\beta}\eta^{\beta}(t)\tilde{a}_2(t)v(g_1(t)) \leq 0.$$

(**10**)

Using $(\mathbf{C}_4)$, we obtain that:

$$\begin{aligned} w(t) &= \eta(t)V(t) + v(t) \\ &\le -\gamma_0\delta_0 v(t) + v(t) \\ &= (1-\gamma_0\delta_0)v(t), \end{aligned}$$

which with (10) gives:

$$w'(t) + \frac{c_0^\beta}{g_0^*}(w(g_0(t)))' + \frac{2^{1-\beta}}{\beta(1-\gamma_0\delta_0)}\eta^\beta(t)\tilde{a}_2(t)w(g_1(t)) \le 0.$$

(**11**)

Now, we set:

$$W(t) := w(t) + \frac{c_0^\beta}{g_0^*}w(g_0(t)) > 0.$$

Then, $W(t) \le \tilde{c}_0 w(g_0(t))$, and so, (11) becomes:

which has a positive solution. In view of [2] (Theorem 1), (9) also has a positive solution, which is a contradiction.

The proof is complete.

Corollary 1. Assume that $g_1(t) \le g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If:

$$\lim_{t\to\infty}\sup \int_{t_1}^{t}\left(\frac{2^{1-\beta}}{\beta}\eta^\beta(\mu)\tilde{a}_2(\mu)\frac{\eta^{\gamma_0\delta_0}(g_1(\mu))}{\eta^{\gamma_0\delta_0}(g_0(\mu))} - \frac{\hat{c}_0}{4}\frac{1}{a_0^{1/\beta}(g_0(\mu))\eta(\mu)}\right)\mathrm{d}\mu = \infty,$$

(**12**)

then every solution of (1) is oscillatory.

Proof. It follows from Theorem 2 in [3] that the condition (12) implies the oscillation of (9).

Next, by introducing two Riccati substitution, we obtain a new oscillation criterion for (1).

Theorem 7. Assume that $g_1(t) \le g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If:

$$\lim_{t\to\infty}\sup \int_{t_1}^{t}\left(\frac{2^{1-\beta}}{\beta}\eta^\beta(\mu)\tilde{a}_2(\mu)\frac{\eta^{\gamma_0\delta_0}(g_1(\mu))}{\eta^{\gamma_0\delta_0}(g_0(\mu))} - \frac{\hat{c}_0}{4}\frac{1}{a_0^{1/\beta}(g_0(\mu))\eta(\mu)}\right)\mathrm{d}\mu = \infty,$$

(**13**)

then every solution of (1) is oscillatory.

Proof. Assume the contrary, that (1) has a solution $u \in U^+$. Then, we have that $u(t)$, and $u(g_1(t))$ are positive for $t \ge t_1$, for some $t_1 \ge t_0$. From Lemmas 1 and 2, we have that $(\mathbf{C}_1)$ – $(\mathbf{C}_4)$ hold for $t \ge t_1$.

Now, we define the functions:

$$\Theta_1 := \frac{V}{v}, \tag{1}$$

and:

$$\Theta_2 := \frac{V\circ g_0}{v\circ g_0}. \tag{2}$$

Then, Θ_1 and Θ_2 are negative for $t \geq t_1$. From $(\mathbf{C}_4)$, we obtain:

$$\frac{v \circ g_1}{\eta^{\gamma_0 \delta_0} \circ g_1} \geq \frac{v \circ g_0}{\eta^{\gamma_0 \delta_0} \circ g_0} \geq \frac{v}{\eta^{\gamma_0 \delta_0}}.$$

Hence,

$$\begin{aligned}
\Theta_1' &= \frac{V'}{v} - \frac{V}{v^2} v' = \frac{V'}{v \circ g_1} \frac{v \circ g_1}{v} - \frac{1}{a^{1/\beta}} \left(\frac{V}{v} \right)^2 \\
&\leq \frac{\eta^{\gamma_0 \delta_0} \circ g_1}{\eta^{\gamma_0 \delta_0}} \frac{V'}{v \circ g_1} - \frac{1}{a^{1/\beta}} \Theta_1^2, \\
&\leq \frac{\eta^{\gamma_0 \delta_0} \circ g_1}{\eta^{\gamma_0 \delta_0} \circ g_0} \frac{V'}{v \circ g_1} - \frac{1}{a^{1/\beta}} \Theta_1^2,
\end{aligned}$$

and:

$$\begin{aligned}
\Theta_2' &= \frac{(V \circ g_0)'}{v \circ g_0} - \frac{V \circ g_0}{(v \circ g_0)^2} (v' \circ g_0) g_0' \\
&= \frac{(V \circ g_0)'}{v \circ g_1} \frac{v \circ g_1}{v \circ g_0} - \frac{g_0'}{a^{1/\beta} \circ g_0} \left(\frac{V \circ g_0}{v \circ g_0} \right)^2 \\
&\leq \frac{\eta^{\gamma_0 \delta_0} \circ g_1}{\eta^{\gamma_0 \delta_0} \circ g_0} \frac{(V \circ g_0)'}{v \circ g_1} - \frac{g_0^*}{(a^{1/\beta} \circ g_0)} \Theta_2^2.
\end{aligned}$$

Then:

$$\eta(t) \Theta_1'(t) - \eta(t) \frac{\eta^{\gamma_0 \delta_0}(g_1(t))}{\eta^{\gamma_0 \delta_0}(g_0(t))} \frac{V'(t)}{v(g_1(t))} + \frac{\eta(t)}{a^{1/\beta}(t)} \Theta_1^2(t) \leq 0,$$

(**14**)

and:

$$\begin{aligned}
0 &\geq \eta(g_0(t)) \Theta_2'(t) - \eta(g_0(t)) \frac{\eta^{\gamma_0 \delta_0}(g_1(t))}{\eta^{\gamma_0 \delta_0}(g_0(t))} \frac{(V(g_0(t)))'}{v(g_1(t))} + \frac{g_0^* \eta(g_0(t))}{a^{1/\beta}(g_0(t))} \Theta_2^2(t) \\
&\geq \eta(g_0(t)) \Theta_2'(t) - \eta(t) \frac{\eta^{\gamma_0 \delta_0}(g_1(t))}{\eta^{\gamma_0 \delta_0}(g_0(t))} \frac{(V(g_0(t)))'}{v(g_1(t))} + \frac{g_0^* \eta(g_0(t))}{a^{1/\beta}(g_0(t))} \Theta_2^2(t).
\end{aligned}$$

(**15**)

Combining (14) and (15), we obtain:

$$\begin{aligned}
0 &\geq \eta(g_0(t)) \Theta_2'(t) - \eta(g_0(t)) \frac{\eta^{\gamma_0 \delta_0}(g_1(t))}{\eta^{\gamma_0 \delta_0}(g_0(t))} \frac{(V(g_0(t)))'}{v(g_1(t))} + \frac{g_0^* \eta(g_0(t))}{a^{1/\beta}(g_0(t))} \Theta_2^2(t) \\
&\geq \eta(g_0(t)) \Theta_2'(t) - \eta(t) \frac{\eta^{\gamma_0 \delta_0}(g_1(t))}{\eta^{\gamma_0 \delta_0}(g_0(t))} \frac{(V(g_0(t)))'}{v(g_1(t))} + \frac{g_0^* \eta(g_0(t))}{a^{1/\beta}(g_0(t))} \Theta_2^2(t).
\end{aligned}$$

Integrating this inequality from t_1 to *t*, we have:

$$\begin{aligned}
0 \geq\ & \eta(t) \Theta_1(t) - \eta(t_1) \Theta_1(t_1) + \int_{t_1}^{t} \left(a_0^{-1/\beta}(\mu) \Theta_1(\mu) + \frac{\eta(\mu)}{a^{1/\beta}(\mu)} \Theta_1^2(t) \right) \mathrm{d}\mu \\
& + \frac{c_0^\beta}{g_0^*} \left(\eta(g_0(t)) \Theta_2(t) - \eta(g_0(t_1)) \Theta_2(t_1) \right) \\
& + \frac{c_0^\beta}{g_0^*} \left(\int_{t_1}^{t} a_0^{-1/\beta}(g_0(t)) \Theta_2(\mu) + \frac{g_0^* \eta(g_0(\mu))}{a^{1/\beta}(g_0(\mu))} \Theta_2^2(\mu) \right) \mathrm{d}\mu \\
& + \frac{2^{1-\beta}}{\beta} \int_{t_1}^{t} \eta^\beta(\mu) \tilde{a}_2(\mu) \frac{\eta^{\gamma_0 \delta_0}(g_1(\mu))}{\eta^{\gamma_0 \delta_0}(g_0(\mu))} \mathrm{d}\mu.
\end{aligned}$$

From $(\mathbf{C}_2)$, we obtain $\eta(t)\Theta_1(t) \geq -1$. Therefore,

where:

$$K := \eta(t_1)\Theta_1(t_1) + \frac{c_0^\beta}{g_0^*}\eta(g_0(t_1))\Theta_2(t_1) + \left(1 + \frac{c_0^\beta}{g_0^*}\right).$$

Since $\eta'(t) < 0$ and $a'(t) \geq 0$, we find:

Taking $\limsup_{t\to\infty}$ and using (13), we arrive at a contradiction.

The proof is complete.

2.3. Applications and Discussion

Remark 1. It is easy to see that the previous works that dealt with the noncanonical case required either $a_1(t) < 1$ or $a_1(t) < \eta(t)/\eta(g_0(t))$. Since η is decreasing and $g_0(t) \leq t$, we have that $\eta(g_0(t)) \geq \eta(t)$. Then, the results of these works only apply when $a_1(t) \in (0,1)$.

Example 1. Consider the DDE:

$$\left(t^2\left(u(t) + a_1^* u(\lambda t)\right)\right)' + a_2^* u(\kappa t) = 0,$$

(**16**)

where $t \geq 1,\ \overset{*}{a_1} > 0,$ and $\kappa < \lambda \in (0,1)$. By choosing $\delta_0 = \overset{*}{a_2},$ the condition (12) becomes:

$$a_2^* \ln\frac{\lambda}{\kappa} > \frac{\lambda + a_1^* - \lambda a_2^*}{e\lambda}.$$

(**17**)

Using Corollary 1, Equation (16) is oscillatory if (17) holds.

Remark 2. To apply Theorems 3 and 4 on (16), we must stipulate that $\overset{*}{a_1} < 1$. Let a special case of (16), namely,

$$\left(t^2\left(u(t) + a_1^* u\left(\frac{t}{2}\right)\right)\right)' + a_2^* u(\kappa t) = 0,$$

A simple computation shows that (16) is oscillatory if:

$$a_2^*(1 - 2a_1^*) > \frac{1}{4}\ (using\ \ Theorem\ \ 3)$$

(**18**)

or:

$$a_2^*(1 - 2a_1^*) > 1\ \ (using\ \ Theorem\ \ 4)$$

(**19**)

or:

$$a_2^*(1 - 2a_1^*)\ln\frac{1}{\kappa} > \frac{1}{e}\ (using\ \ Theorem\ \ 5).$$

(**20**)

Consider the following most specific special case:

$$\left(t^2\left(u(t) + \frac{2}{5}u\left(\frac{t}{2}\right)\right)\right)' + \frac{4}{5}u\left(\frac{t}{4}\right) = 0.$$

(**21**)

Note that (18)–(20) fail to apply. However, (17) reduces to:

$$\frac{4}{5}\ln 2 > \frac{1}{e}.$$

which ensures the oscillation of (21).

References

1. Baculikova, B.; Dzurina, J. Oscillation theorems for second-order nonlinear neutral differential equations. Comput. Math. Appl. 2011, 62, 4472–4478.
2. Philos, C. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. Arch. Math. (Basel) 1981, 36, 168–178.
3. Kitamura, Y.; Kusano, T. Oscillation of first-order nonlinear differential equations with deviating arguments. Proc. Amer. Math. Soc. 1980, 78, 64–68.