Revisiting Lorenz's Error Growth Models: Insights and Applications

Subjects: Meteorology & Atmospheric Sciences

Contributor: Bo-Wen Shen

This entry examines Lorenz's error growth models with quadratic and cubic hypotheses, highlighting their mathematical connections to the non-dissipative Lorenz 1963 model. The quadratic error growth model is the logistic ordinary differential equation (ODE) with a quadratic nonlinear term, while the cubic model is derived by replacing the quadratic term with a cubic one. A variable transformation shows that the cubic model can be converted to the same form as the logistic ODE. The relationship between the continuous logistic ODE and its discrete version, the logistic map, illustrates chaotic behaviors, demonstrating computational chaos with large time steps. A variant of the logistic ODE is proposed to show how finite predictability horizons can be determined, emphasizing the continuous dependence on initial conditions (CDIC) related to stable and unstable asymptotic values. This review also presents the mathematical relationship between the logistic ODE and the non-dissipative Lorenz 1963 model.

Lorenz models error growth logistic ODE logistic map CDIC predictability horizons

The entry offers a concise overview of the mathematical framework for Lorenz error growth models (Lorenz 1969a [1]) and their connections to other relevant systems (e.g., the non-dissipative Lorenz model, Shen 2018, 2020 [2][3]). In the 1960s, Lorenz utilized the first-order ordinary differential equations (ODEs) incorporating quadratic or cubic nonlinear terms to study root mean square (RMS) errors, aiding in the estimation of doubling times and thus predictability horizons (Lorenz 1969a, b [1][4]; Shen et al., 2024 [5]). The model with the quadratic term is commonly referred to as the logistic ODE. In contrast, the error growth model with a cubic term can be transformed into the logistic ODE (with a quadratic term). It is important to distinguish this error growth model from Lorenz's widely recognized 1969 multiscale model in meteorology (Lorenz 1969c [6]; Shen et al., 2024 [5]).

Since the 1960s, the logistic ODE, the continuous form of the logistic equation, has been extensively used to assess RMS forecast errors in ensemble forecasts (e.g., Lorenz 1969a, 1982, 1996 [1][7][8]; Nicolis 1992 [9]; Kalnay 2003 [10]; Zhang et al. 2019 [11]). Historically, the logistic ODE has also been used to analyze population growth, assuming that the growth rate of a population is proportional to both the existing population and the available resources (represented by the growth rate σ

in Equation (1)). The logistic ODE-based model offers a more realistic portrayal of population growth compared to simple exponential models. This model has also been adapted to study the dynamics of recovered individuals during the COVID-19 pandemic (Postnikov, 2020 [12]; Paxson and Shen 2022 [13]).

The logistic map, a discrete form of the logistic equation, stands as the most straightforward model for demonstrating chaos, as recognized in the works of Lorenz (1964, 1969d) [14][15], May (1976) [16], and Li and Yorke (1975) [17]. This map exhibits both periodic and chaotic behaviors, heavily influenced by a system parameter that acts as a forcing term. Initially introduced by Lorenz in 1964, the map was used to describe the transition from regular to irregular solutions in dynamic systems. Robert May (1976) [16] utilized this map to investigate population dynamics in biological contexts, while Li and Yorke (1975) [17] employed it to demonstrate the concept of “Period Three Implies Chaos”, where the term “chaos” was initially introduced to describe irregular solutions. Consequently, the logistic map has become an influential tool in studying chaos and complexity within dynamical systems and was recognized by Stewart (2013) [18] as one of the 17 equations that changed the world.

This entry explores both the continuous and discrete versions of the logistic equation, providing a mathematical perspective on Lorenz models. It highlights a crucial, yet often overlooked, aspect of continuous dependence on initial conditions (CDIC). This concept is demonstrated by showing how the predictability horizon, as derived from the logistic ODE, can be extended progressively by refining the initial conditions. The structure of the entry is as follows: After presenting the equations for the Logistic ODE, this entry analyzes its error growth rates and its linkage to the Logistic map. The Logistic ODE and map are employed to illustrate CDIC and irregular solutions, respectively. This entry also explores the connection between the logistic ODE and the non-dissipative Lorenz model (Lorenz 1963 [19]; Shen 2020, 2023 [3][20]), and finally discusses the finite predictability horizons in a variant of the logistic ODE. Concluding remarks are presented at the end.

References

1. Shen, B.-W. On periodic solutions in the non-dissipative Lorenz model: The role of the nonlinear feedback loop. Tellus A 2018, 70, 1471912.
2. Shen, B.-W. Homoclinic Orbits and Solitary Waves within the non-dissipative Lorenz Model and KdV Equation. Int. J. Bifurc. Chaos 2020, 30, 2050257.
3. Lorenz, E.N. Studies of Atmospheric Predictability. Final Report, February, Statistical Forecasting Project; Air Force Research Laboratories, Office of Aerospace Research, USAF: Bedford, MA, USA, 1969; p. 145.
4. Shen, B.-W.; Pielke, R.A.; Zeng, X. Exploring the Origin of the Two-Week Predictability Limit: A Revisit of Lorenz's Predictability Studies in the 1960s. Atmosphere, 2024; in press.
5. Lorenz, E.N. The predictability of a flow which possesses many scales of motion. Tellus 1969, 21, 289–307.
6. Lorenz, E.N. Atmospheric predictability experiments with a large numerical model. Tellus 1982, 34, 505–513.

7. Lorenz, E.N. Predictability—A problem Partly Solved. In Proceedings of the Seminar on Predictability, Reading, UK, 4–8 September 1995; ECMWF: Reading, UK, 1996; Volume 1.
8. Nicolis, C. Probabilistic aspects of error growth in atmospheric dynamics. Q. J. R. Meteorol. Soc. 1992, 118, 553–568.
9. Kalnay, E. Atmospheric Modeling, Data Assimilation and Predictability; Cambridge: New York, NY, USA, 2003; p. 369.
10. Zhang, F.; Sun, Y.Q.; Magnusson, L.; Buizza, R.; Lin, S.-J.; Chen, J.-H.; Emanuel, K. What is the predictability limit of midlatitude weather? J. Atmos. 2019, 76, 1077–1091.
11. Postnikov, E.B. Estimation of COVID-19 dynamics “on a back-of-envelope”: Does the simplest SIR model provide quantitative parameters and predictions? Chaos Solitons Fractals 2020, 135, 109841.
12. Paxson, W.; Shen, B.-W. A KdV-SIR Equation and Its Analytical Solutions for Solitary Epidemic Waves. Int. J. Bifurc. Chaos 2022, 32, 13.
13. Lorenz, E.N. The problem of deducing the climate from the governing equations. Tellus 1964, 16, 1–11.
14. Lorenz, E.N. How much better can weather prediction become? MIT Technol. Rev. 1969, 1969, 39–49.
15. May, R.M. Simple Mathematical Models with very Complicated Dynamics. Nature 1976, 261, 459–467.
16. Li, T.-Y.; Yorke, J.A. Period Three Implies Chaos. Am. Math. Mon. 1975, 82, 985–992.
17. Stewart, I. Seventeen Equations That Changed the World; Profile Book: London, UK, 2013; p. 342.
18. Lorenz, E.N. Deterministic nonperiodic flow. J. Atmos. Sci. 1963, 20, 130–141.
19. Shen, B.-W. A Review of Lorenz’s Models from 1960 to 2008. Int. J. Bifurc. Chaos 2023, 33, 2330024.