Gauss' Method

In orbital mechanics (subfield of celestial mechanics), Gauss's method is used for preliminary orbit determination from at least three observations (more observations increases the accuracy of the determined orbit) of the orbiting body of interest at three different times. The required information are the times of observations, the position vectors of the observation points (in Equatorial Coordinate System), the direction cosine vector of the orbiting body from the observation points (from Topocentric Equatorial Coordinate System) and general physical data. Carl Friedrich Gauss developed important mathematical techniques (summed up in Gauss's methods) which were specifically used to determine the orbit of Ceres. The method shown following is the orbit determination of an orbiting body about the focal body where the observations were taken from, whereas the method for determining Ceres' orbit requires a bit more effort because the observations were taken from Earth while Ceres orbits the Sun.

Keywords: orbital mechanics ; orbit determination ; ceres

1. Observer Position Vector

The observer position vector (in Equatorial Coordinate System) of the observation points can be determined from the latitude and local sidereal time (from Topocentric Coordinate System) at the surface of the focal body of the orbiting body (for example, the Earth) via either:

$$\mathbf{R_n} = \left[\frac{R_e}{\sqrt{1 - (2f - f^2)\sin^2\phi_n}} + H_n \right] \cos\phi_n (\cos\theta_n \hat{\mathbf{I}} + \sin\theta_n \hat{\mathbf{J}}) + \left[\frac{R_e(1-f)^2}{\sqrt{1 - (2f - f^2)\sin^2\phi_n}} + H_n \right] \sin\phi_n \hat{\mathbf{K}}$$

or

$$\mathbf{R_n} = R_e \cos\phi_n' \cos\theta_n \hat{\mathbf{I}} + R_e \cos\phi_n' \sin\theta_n \hat{\mathbf{J}} + R_e \sin\phi_n' \hat{\mathbf{K}}$$

where,

- $\mathbf{R_n}$ is the respective observer position vector (in Equatorial Coordinate System)
- R_e is the equatorial radius of the body (e.g., Earth's Re is 6,378 km)
- f is the oblateness (or flattening) of the body (e.g., Earth's f is 0.003353)
- ϕ_n is the respective geodetic latitude
- ϕ'_n is the respective geocentric latitude
- H_n is the respective altitude
- θ_n is the respective local sidereal time

2. Orbiting Body Direction Cosine Vector

The orbiting body direction cosine vector can be determined from the right ascension and declination (from Topocentric Equatorial Coordinate System) of the orbiting body from the observation points via:

$$\hat{\rho}_{\mathbf{n}} = \cos\delta_n \cos\alpha_n \hat{\mathbf{I}} + \cos\delta_n \sin\alpha_n \hat{\mathbf{J}} + \sin\delta_n \hat{\mathbf{K}}$$

where,

- $\hat{\rho}_n$ is the respective unit vector in the direction of the position vector ρ (from observation point to orbiting body in Topocentric Equatorial Coordinate System)
- δ_n is the respective declination
- α_n is the respective right ascension

3. Gauss's Method of Preliminary Orbit Determinations Algorithm

The following equations are shown without derivation because a basic background understanding is necessary. However, a brief explanation will be given to provide an idea of the derivation process.

The initial derivation begins with vector addition to determine the orbiting body's position vector. Then based on the conservation of angular momentum and Keplerian orbit principles (which states that an orbit lies in a two dimensional plane in three dimensional space), a linear combination of said position vectors is established. Also, the relation between a body's position and velocity vector by Lagrange coefficients is used which results in the use of said coefficients. Then with vector manipulation and algebra, the following equations were derived. For detailed derivation, refer to Curtis.[1]

NOTE: Gauss's method is a preliminary orbit determination, with emphasis on preliminary. The approximation of the Lagrange coefficients and the limitations of the required observation conditions (i.e., insignificant curvature in the arc between observations, refer to Gronchi[2] for more details) causes inaccuracies. There are ways of increasing the accuracy of Gauss's method such as increasing the accuracy of the Lagrange coefficients. This can be done by solving the universal Kepler's equation. Another way to increase the accuracy is through more observations. There are techniques/methods available that can be used but why not use Gauss's own method, least squares method (still popularly used today).

3.1. Step 1

Calculate time intervals, subtract the times between observations:

$$\tau_1 = t_1 - t_2$$
$$\tau_3 = t_3 - t_2$$
$$\tau = t_3 - t_1$$

where

τ_n is the time interval
t_n is the respective observation time

3.2. Step 2

Calculate cross products, take the cross products of the observational unit direction (order matters):

$$\mathbf{p_1} = \hat{\rho}_2 \times \hat{\rho}_3$$
$$\mathbf{p_2} = \hat{\rho}_1 \times \hat{\rho}_3$$
$$\mathbf{p_3} = \hat{\rho}_1 \times \hat{\rho}_2$$

where

$\mathbf{p_n}$ is the respective cross product vector

$\hat{\rho}_n$ is the respective unit vector

3.3. Step 3

Calculate common scalar quantity (scalar triple product), take the dot product of the first observational unit vector with the cross product of the second and third observational unit vector:

$$D_0 = \hat{\rho}_1 \cdot \mathbf{p_1} = \hat{\rho}_1 \cdot (\hat{\rho}_2 \times \hat{\rho}_3)$$

where
D_0 is the common scalar triple product
$\mathbf{p_n}$ is the respective cross product vector
$\hat{\rho}_n$ is the respective unit vector

3.4. Step 4

Calculate nine scalar quantities (similar to step 3):

$$D_{11} = \mathbf{R_1} \cdot \mathbf{p_1} \qquad D_{12} = \mathbf{R_1} \cdot \mathbf{p_2} \qquad D_{13} = \mathbf{R_1} \cdot \mathbf{p_3}$$

$$D_{21} = \mathbf{R_2} \cdot \mathbf{p_1} \qquad D_{22} = \mathbf{R_2} \cdot \mathbf{p_2} \qquad D_{23} = \mathbf{R_2} \cdot \mathbf{p_3}$$

$$D_{31} = \mathbf{R_3} \cdot \mathbf{p_1} \qquad D_{32} = \mathbf{R_3} \cdot \mathbf{p_2} \qquad D_{33} = \mathbf{R_3} \cdot \mathbf{p_3}$$

where

D_{mn} is the respective scalar quantities

$\mathbf{R_m}$ is the respective observer position vector

$\mathbf{p_n}$ is the respective cross product vector

3.5. Step 5

Calculate scalar position coefficients:

$$A = \frac{1}{D_0}\left(-D_{12}\frac{\tau_3}{\tau} + D_{22} + D_{32}\frac{\tau_1}{\tau}\right)$$

$$B = \frac{1}{6D_0}\left[D_{12}\left(\tau_3^2 - \tau^2\right)\frac{\tau_3}{\tau} + D_{32}\left(\tau^2 - \tau_1^2\right)\frac{\tau_1}{\tau}\right]$$

$$E = \mathbf{R_2} \cdot \hat{\rho}_2$$

where

A, B, and E are scalar position coefficients

D_0 is the common scalar quantity

D_{mn} is the respective scalar quantities

τ_n is the time interval

$\mathbf{R_n}$ is the respective observer position vector

$\hat{\rho}_n$ is the respective unit vector

3.6. Step 6

Calculate the squared scalar distance of the second observation, by taking the dot product of the position vector of the second observation:

$$R_2{}^2 = \mathbf{R_2} \cdot \mathbf{R_2}$$

where

R_2^2 is the squared distance of the second observation

$\mathbf{R_2}$ is the position vector of the second observation

3.7. Step 7

Calculate the coefficients of the scalar distance polynomial for the second observation of the orbiting body:

$$a = -\left(A^2 + 2AE + R_2{}^2\right)$$

$$b = -2\mu B(A + E)$$

$$c = -\mu^2 B^2$$

where

a, b and c are coefficients of the scalar distance polynomial for the second observation of the orbiting body

A,B and E are scalar position coefficients

μ is the gravitational parameter of the focal body of the orbiting body

3.8. Step 8

Find the root of the scalar distance polynomial for the second observation of the orbiting body:

$$r_2{}^8 + ar_2{}^6 + br_2{}^3 + c = 0$$

where

r_2 is the scalar distance for the second observation of the orbiting body (it and its vector, $\mathbf{r_2}$, are in the Equatorial Coordinate System)

a,b and c are coefficients as previously stated

Various methods can be used to find the root, a suggested method is the Newton-Raphson method. The root must be physically possible (i.e., not negative nor complex) and if multiple roots are suitable, each must be evaluated and compared to any available data to confirm their validity.

3.9. Step 9

Calculate the slant range, the distance from the observer point to the orbiting body at their respective time:

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$$\rho_2 = A + \frac{\mu B}{r_2{}^3}$$

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where
ρ_n is the respective slant range (it and its vector, $\boldsymbol{\rho_n}$, are in the Topocentric Equatorial Coordinate System)
D_0 is the common scalar quantity
D_{mn} is the respective scalar quantities
$\tau_{(n)}$ is the time interval
r_2 is the scalar distance for the second observation of the orbiting body
μ is the gravitational parameter of the focal body of the orbiting body

3.10. Step 10

Calculate the orbiting body position vectors, by adding the observer position vector to the slant direction vector (which is the slant distance multiplied by the slant direction vector):

$$\mathbf{r_1} = \mathbf{R_1} + \rho_1\hat{\rho}_1$$

$$\mathbf{r_2} = \mathbf{R_2} + \rho_2\hat{\rho}_2$$

$$\mathbf{r_3} = \mathbf{R_3} + \rho_3\hat{\rho}_3$$

where
$\mathbf{r_n}$ is the respective orbiting body position vector (in Equatorial Coordinate System)
$\mathbf{R_n}$ is the respective observer position vector
ρ_n is the respective slant range
$\hat{\rho}_n$ is the respective unit vector

3.11. Step 11

Calculate the Lagrange coefficients:

$$f_1 \approx 1 - \frac{1}{2}\frac{\mu}{r_2{}^3}\tau_1{}^2$$

$$f_3 \approx 1 - \frac{1}{2}\frac{\mu}{r_2{}^3}\tau_3{}^2$$

$$g_1 \approx \tau_1 - \frac{1}{6}\frac{\mu}{r_2{}^3}\tau_1{}^3$$

$$g_3 \approx \tau_3 - \frac{1}{6}\frac{\mu}{r_2{}^3}\tau_3{}^3$$

where,
f_1, f_3, g_1 and g_3 are the Lagrange coefficients (these are just the first two terms of the series expression based on the assumption of small time interval)
μ is the gravitational parameter of the focal body of the orbiting body
r_2 is the scalar distance for the second observation of the orbiting body
$\tau_{(n)}$ is the time interval

3.12. Step 12

Calculate the velocity vector for the second observation of the orbiting body:

$$\mathbf{v_2} = \frac{1}{f_1 g_3 - f_3 g_1}(-f_3 \mathbf{r_1} + f_1 \mathbf{r_3})$$

where

$\mathbf{v_2}$ is the velocity vector for the second observation of the orbiting body (in Equatorial Coordinate System)

f_1, f_3, g_1 and g_3 are the Lagrange coefficients

$\mathbf{r_n}$ is the respective orbiting body position vector

3.13. Step 13

The orbital state vectors have now been found, the position (r2) and velocity (v2) vector for the second observation of the orbiting body. With these two vectors, the orbital elements can be found and the orbit determined.

References

1. Curtis, Howard D. Orbital Mechanics for Engineering Students. Oxford: Elsevier Butterworth-Heinemann, 2005. Print.

2. Gronchi, Giovanni F.. "Classical and modern orbit determination for asteroids." Proceedings of the International Astronomical Union2004.IAUC196 (2004): 1-11. Print. http://copernico.dm.unipi.it/~gronchi/PDF/gronchi_discuss.pdf